1. In this problem, we will use Taylor approximation to approximate the integral

\[ I = \int_1^2 \frac{e^x - 1}{x} \, dx. \]

Let us denote \( f(x) = \frac{e^x - 1}{x} \).

(a) Derive a formula for the \( n \)’th Taylor polynomial about \( x_0 = 0 \), called \( p_n(x) \), of \( f \). Use the summation symbol \( \Sigma \) to write \( p_n(x) \).

*Hint: use the Taylor approximation of the function \( e^x \).*

(b) Write the integral \( I_n = \int_1^2 p_n(x) \, dx \) using \( \Sigma \) symbol and without integral signs.

(c) How large should \( n \) be so that \( I_n \) approximates \( I \) with an error less than \( \epsilon = 10^{-5} \)?

(d) With a value of \( n \) found in Part (c), write a Matlab code to compute \( I_n \). Matlab has a built-in function called ‘int’ to compute approximately \( I \). Try the following:

```matlab
format long
f = @(x) (exp(x)-1)./x
integral(f,1,2)
```

Double check if \( I_n \) indeed approximates \( I \) with error less than \( 10^{-5} \).

2. Let us consider the following toy model of the IEEE double precision floating-point format. This toy model makes it simpler to demonstrate how addition and multiplication of floating-point numbers work.

The sequence of 8 bits

\[
\begin{array}{cccccccc}
\text{c}_0 & \text{b}_1 & \text{b}_2 & \text{b}_3 & \text{a}_1 & \text{a}_2 & \text{a}_3 \\
\text{sign part} & \text{exponent part} & \text{mantissa part}
\end{array}
\]

represents a number \( x = \sigma \cdot \bar{x} \cdot 2^e \) where \( \sigma, \bar{x}, e \) are determined as follows:

\[
\sigma = \begin{cases} 
1 & \text{if } \text{c}_0 = 0, \\
-1 & \text{if } \text{c}_0 = 1,
\end{cases}
\]

\[
E = (b_1b_2b_3b_4)_2
\]

- If \( 1 \leq E \leq 14 \) then

\[
e = E - 7,
\]

\[
\bar{x} = (1.a_1a_2a_3)_2
\]

- If \( E = 0 \) then \( e = -6 \) and \( \bar{x} = (0.a_1a_2a_3)_2 \).
- If \( E = 15 \) then \( x = \pm \infty \) (depending on the sign \( \sigma \)).

(a) Find the dynamic range and machine epsilon of this floating-point number format.

(b) What numbers are represented by the bit sequences 11001001, 00000000, 11111000 ?
3. There are only 256 different sequences of 8 bits. Thus, the sequence of 8 bits in Problem 2 cannot represent precisely every real number. It can represent precisely only 254 real numbers and ±∞. However, any real number can be represented approximately by a bit sequence. The principle is simple: given a real number \( x \), we look for the number \( y \) among those 256 numbers that is closest to \( x \). Then \( x \) is represented by the bit sequence that represents \( y \).

The method is as follows:

- Write \( x \) is binary form. For example, \( 6.3 = (110.010011001 \ldots)_2 \).
- Shift the binary point to the form \( 1.c_1c_2c_3 \ldots \) by choosing an exponent \(-6 \leq e \leq 7\). For example, \( 6.3 = (1.10010011001 \ldots)_2 \times 2^2 \).
- Round the mantissa to 3 digits after the dot. For example, \( 6.3 \approx (1.101)_2 \times 2^2 \).
- Find the value of \( \sigma, \bar{x}, e \). For example, these values in the case \( x = 6.3 \) are \( \sigma = 1 \), \( \bar{x} = (1.101)_2 \) and \( e = 2 \). The bit sequence that represents 6.3 is therefore 01001101.

Note that in the second step, it may be impossible to choose \( e \) between \(-6 \) and 7. An example is when \( e > 7 \). In this case, the number is “too big” and is approximated by ±∞ (depending on the sign \( \sigma \)). Another example is when \( e < -6 \). In this case, one will shift the binary point one digit to the left to get the form \((0.1c_1c_2c_3 \ldots)_2 \). The new exponent is now \( e + 1 \). If the new exponent is equal to \(-6 \) then one proceeds to Step 3 and 4. If the new exponent is still less than \(-6 \), the number \( x \) is “too close to zero” and thus is approximated by 0.

(a) Represent the decimal numbers 1, 5.5, 12.9, 1000, 0.0001 in the floating-point format \( x = \sigma \cdot \bar{x} \cdot 2^e \) and bit sequence described in Problem 2.

(b) Find the smallest number larger than 5.5 that can be represented precisely by the floating-point format in Problem 2. The same question for 12.9 and 100.25.