Denote \( I = \int_0^1 \frac{1}{1+4x^2} \, dx \).

**Problem 1.**

Find the exact value of \( I \).

Solution

\[
\int_0^2 \frac{1}{1+4x^2} \, dx = \frac{1}{2} \arctan(2x) \bigg|_0^1 = \frac{1}{2} (\arctan(2) - \arctan(0)) = \frac{\arctan(2)}{2}
\]

With a calculator, \( I \approx 0.231823804500403 \).

**Problem 2.**

For a generic positive integer \( n \) we take \( n + 1 \) equally spaced sample points indexed by \( x_0, x_1, \ldots, x_n \) on the interval \([0, 1]\). Denote by \( L_n, R_n, M_n, T_n \) the Riemann sums corresponding to the left-point, right-point, midpoint, and trapezoid rule. Use sigma notation to write a formula for each \( L_n, R_n, M_n, T_n \).

Solution

Set \( x_i = \frac{i}{n} \).

\[
L_n = \sum_{i=0}^{n-1} \frac{1}{n} f(x_i) = \sum_{i=0}^{n-1} \frac{1}{n} \frac{1}{1+4x_i^2} = \sum_{i=0}^{n-1} \frac{1}{n \left( 1 + \left( \frac{2i}{n} \right)^2 \right)} = \sum_{i=0}^{n-1} \frac{n}{n^2 + 4i^2}
\]

\[
R_n = \sum_{i=1}^{n} \frac{1}{n} f(x_i) = \sum_{i=1}^{n} \frac{n}{n^2 + 4i^2}
\]

Note that the indexing has changed between \( L_n \) and \( R_n \).

\[
M_n = \sum_{i=0}^{n-1} \frac{1}{n} f\left( \frac{x_i + x_{i+1}}{2} \right) = \sum_{i=0}^{n-1} \frac{1}{n} \frac{1}{1 + \left( \frac{2i+1}{n} \right)^2} = \sum_{i=0}^{n-1} \frac{n + \frac{1}{n} (2i + 1)}{n^2 + (2i + 1)^2} = \sum_{i=0}^{n-1} \frac{n}{n^2 + (2i + 1)^2}
\]

\[
T_n = \sum_{i=0}^{n-1} \frac{1}{n} \frac{f(x_i) + f(x_{i+1})}{2} = \sum_{i=0}^{n-1} \frac{1}{2n} \left( \frac{1}{1 + \left( \frac{2i}{n} \right)^2} + \frac{1}{1 + \left( \frac{2i+2}{n} \right)^2} \right)
\]

**Problem 3.**

Which of these three methods gives the best approximation of \( I \) when \( n = 4 \)?

Solution

We can use the above formulas to compute the approximations, then find the error bounds.

\[
|L_4 - I| \approx 0.098348718026032
\]

\[
|R_4 - I| \approx 0.101651281973968
\]

\[
|M_4 - I| \approx 0.101651281973968
\]

\[
|T_4 - I| \approx 0.001651281973968
\]

\( T_4 \) gives the best approximation of the four choices. The left, right, and midpoint approximations are of similar quality.
Problem 4.

Write Matlab code to compute $L_n$, $R_n$, $M_n$, and $T_n$ for $n = 8, 16, 32, 64$.

Solution

We do this in Matlab. For simplicity, we write one script for each method which allows us to easily change $n$ (line 3). Both the approximation and error are printed to the console.

Left point method:

```matlab
a = 0;
b = 1;
n = 4;
xvals = linspace(a,b,n+1); % Generate n+1 points
yvals = objective(xvals);
total = 0;
for ii = 1:n
    total = total + (yvals(ii))*(b-a)/n;
end
disp(total)
I = atan(2)/2;
error = total - I;
disp(abs(error))
function out = objective(in)
    out = 1./(1 + 4.*in.^2);
end
```

Right point method:

```matlab
a = 0;
b = 1;
n = 4;
xvals = linspace(a,b,n+1); % Generate n+1 points
yvals = objective(xvals);
total = 0;
for ii = 1:n
    total = total + (yvals(ii+1))*(b-a)/n;
end
disp(total)
I = atan(2)/2;
error = total - I;
disp(abs(error))
function out = objective(in)
    out = 1./(1 + 4.*in.^2);
end
```

Trapezoidal method:

```matlab
a = 0;
b = 1;
n = 4;
xvals = linspace(a,b,n+1); % Generate n+1 points
yvals = objective(xvals);
total = 0;
for ii = 1:n
    total = total + (yvals(ii) + yvals(ii+1))/2*(b-a)/n;
end
```

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Midpoint method:

```matlab
a = 0;
b = 1;
n = 4;
xvals = linspace(a,b,n+1); % Generate n+1 points
xvals = xvals + 1/n; % Shift xvalues by one half
yvals = objective(xvals);
total = 0;
for ii = 1:n
    total = total + (yvals(ii))*(b-a)/n;
end
disp(total)
I = atan(2)/2;
error = total - I;
disp(abs(error))
function out = objective(in)
    out = 1./(1 + 4.*in.^2);
end
```

**Problem 5.**

Find values of $n$ such that the error for each left point, right point, midpoint, and trapezoidal rule approximations are bounded by $\epsilon = 0.0001$.

**Solution**

We need to find bounds on $K$ and $\tilde{K}$.

\[
|f'(x)| = \frac{8x}{(1+4x^2)^2} \Rightarrow |f'(x)| \leq \frac{8(1)}{(1+4(0)^2)^2} = 8, \quad x \in [0,1]
\]

So $K \leq 8$. We can solve the constrained optimization problem on $[0,1]$ by finding roots of $f''$ on $[0,1]$.

\[
|f''(x)| = \left| \frac{128x^2}{(1+4x^2)^3} - \frac{8}{(1+4x^2)^4} \right| = \frac{96x^2 - 8}{(1+4x^2)^3} \leq \frac{96(1)^2 - 8}{(1+4(0)^2)^3} = 88.
\]

Hence, $\tilde{K} \leq 88$.

Now we can compute bounds.

\[
e_n^{(L)} \leq \frac{(b-a)^2}{2n} K \leq \frac{(1-0)^2}{2n} 8 = \frac{4}{n}.
\]

To make sure that $e_n^{(L)} < 10^{-4}$, we only need to require that $4/n < 10^{-4}$. We see that any $n > 40000$ will do it. We have

\[
e_n^{(M)} \leq \frac{(b-a)^3}{24n^2} \tilde{K} \leq \frac{(1-0)^3}{24n^2} 88 = \frac{11 \cdot 1}{3 \cdot n^2}.
\]

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To make sure that $e_n^{(M)} < 10^{-4}$, we only need to require that \( \frac{11}{3} \frac{1}{n^2} < 10^{-4} \). We see that any $n \geq 192$ will do it. This is a much smaller $n$ than in the left or right point approximations.

\[
e_n^{(R)} \leq \frac{(b-a)^3}{12n^2} \tilde{K} \leq \frac{(1-0)^3}{12n^2} 88 = \frac{22}{3} \frac{1}{n^2}.
\]

To make sure that $e_n^{(R)} < 10^{-4}$, we only need to require that \( \frac{22}{3} \frac{1}{n^2} < 10^{-4} \). We see that any $n \geq 271$ will do it. This is still a much smaller $n$ than in the left or right point approximations.