With the bisection method and the Newton's method, we are able to find an approximate root of the equation $f(x) = 0$. This is a rootfinding problem. There is "sister" problem of rootfinding problem, known as fixed point problem.

- $x$ is a root of $f$ if $f(x) = 0$.
- $x$ is a fixed point of $f$ if $f(x) = x$.

The intersection of the graph of $f$ and the $x$-axis are the roots of $f$ (the black dots). The intersection of the graph of $f$ and the line $y = x$ are the fixed points of $f$ (the red dots).

A fixed point problem can be converted into a rootfinding problem, for example,

\[ f(x) - x = 0, \]
\[ \frac{f(x)}{2} - 1 = 0, \]
\[ f(x)^2 - x^3 = 0, \]

Likewise, a root finding problem can be converted into a fixed point problem, for example

\[ f(x) + x = x, \]
\[ 2(f(x) + 1) = x, \]
\[ x - \frac{f(x)}{f'(x)} = x, \quad \text{(Newton's method)} \]
Ex: Let us do the following experiment on the calculator:
Press \( \boxed{5} \), then press \( \boxed{=} \). Then press \( \boxed{\cos} \), \( \boxed{\text{ANS}} \). Then press \( \boxed{=} \) many times. We get a new number each time we press the equal sign.
After a number of times, we get a steady result, which is about 0.7390.
Let us call this number \( a = 0.7390 \). After pressing the equal sign, we should get \( \cos a \). What is shown on the screen is \( a \). Thus,
\[
\cos a = a.
\]
In other words, we have found a fixed point of the function cosine.

Let us take a closer look: we start with the initial value \( x_0 = 1 \).
Then we get \( x_1 = \cos x_0 \) after pressing \( \boxed{=} \). Then we get \( x_2 = \cos x_1 \),
then \( x_3 = \cos x_2 \), ..., In general, we get a sequence \( (x_n) \) which is defined recursively as follows:
\[
\begin{cases} 
  x_0 = 1, \\
  x_{n+1} = \cos x_n.
\end{cases}
\]
If this sequence has a limit, say \( x = \lim x_n \). Then by taking the limit of both sides of the equation \( x_{n+1} = \cos x_n \), we get
\[
x = \cos x.
\]
This confirms the experimental observation that \( x_n \) converges to a fixed point of \( f(x) = \cos x \).
The fixed point method has a very elegant illustration called cobweb diagram.

Starting from \( x_0 \), we draw a vertical line. It intersects the graph of \( f \) at exactly one point. The y-coordinate of
this point is $m$. Then from this intersection point, draw a horizontal line. This line intersects the line $y = x$ at exactly one point. The $x$-coordinate of this point is $x_1$. Then from this intersection point, draw a vertical line. The intersection of this line and the graph of $f$ has $y$-coordinate equal to $m$. Then from the intersection point, draw a horizontal line. The intersection of this line and the line $y = x$ has $x$-coordinate equal to $x_2$. We repeat this process.

One can experiment the fixed point method using cobweb diagram. A helpful applet can be found at the website:

https://www.geogebra.org/m/QJ79IWCL

Ex:

The function $f(x) = 3x - x^2$ has two fixed points $x = 0$ and $x = 2$.

The recursive sequence is $x_{n+1} = f(x_n) = 3x_n - x_n^2$.

With $x_0 = 1.8$, we see that the sequence $x_n$ converges (very slowly) to 2.

Ex: $g(x) = x^3$

This function has two fixed points: $x = 0$ and $x = 1$.

If we choose $x_0 > 1$, we will get a divergence sequence. To be more precise, the sequence goes to infinity as $n \to \infty$. In a loose sense, one can say that this “limit” (the infinity) is a fixed point of $f$ because $g(\infty) = \infty$.

If we choose $0 < x_0 < 1$ then the sequence $x_n$ will converge to 0.
It seems that the sequence will not converge to the fixed point 1 unless the initial point \( x_0 \) is chosen to be exactly 1. In this case, 1 said to be an unstable fixed point, and 0 is said to be a stable fixed point.

A fixed point \( x^* \) is called stable if any choice of \( x_0 \) sufficiently near \( x^* \) will yield a sequence that converges to \( x^* \).

**Ex.**

Solve for a root of \( f(x) = x^3 - 3x + 1 \) using fixed point method. There are many ways to convert the root-finding problem

\[
x^3 - 3x + 1 = 0 \quad (x)
\]

into a fixed point problem. One way is to rewrite the equation as

\[
x^3 - 3x + 1 = x
\]

\[
g(x)
\]

The roots of \( f \) are the fixed points of \( g \). By experimenting on the cobweb plotter, one can observe chaotic behavior of the sequence \( \{x_n\} \).

In other words, the sequence doesn’t seem to converge. The choice of \( g \) is not good.

[Coebweb diagram of \( g \) with \( x_0 = 0.8 \)]
Equation (*) can be written as
\[ x^3 + 1 = 3x \quad \text{or} \quad \frac{1}{3}(x^3 + 1) = x. \]
\[ h(x) \]

We see that the sequence \( x_n \) converges quickly to a fixed point. Thus, the choice of \( h \) is better than \( g \).