1. Approximate the value of $e$ with acceptable error $\epsilon = 10^{-3}$ by using Taylor approximation for the function $f(x) = e^x$ about $x_0 = 0$.

Note that $e = f(1)$. We have

$$f(x) = f'(x) = f''(x) = f^{(3)}(x) = \ldots = e^x$$

and $f(0) = f'(0) = f''(0) = \ldots = 1$.

The $n$th Taylor polynomial of $f$ is

$$p_n(x) = f(0) + \frac{f'(0)}{1!}(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \ldots + \frac{f^{(n)}(0)}{n!}(x-0)^n$$

$$= 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots + \frac{1}{n!}x^n.$$ 

We have

$$f(x) = p_n(x) + \frac{R_n(x)}{x^n}.$$ 

At $x=1$,

$$f(1) = p_n(1) + R_n(1).$$

By Lagrange formula,

$$R_n(1) = \frac{f^{(n+1)}(c)}{(n+1)!} (1-0)^{n+1} = \frac{f^{(n+1)}(c)}{(n+1)!} = \frac{e^c}{(n+1)!},$$

where $c$ is some number between $x=0$ and $x=1$.

Then

$$|R_n(1)| = \frac{e^c}{(n+1)!} < \frac{e}{(n+1)!} < \frac{3}{(n+1)!}$$
To make sure that $|R_n(0)| < 10^{-3}$, we need to choose $n$ sufficiently large such that

$$\frac{3}{(n+1)!} < 10^{-3}.$$  

Using calculator, we can choose $n = 6$. Then

$$e = f(1) \approx P_6(1) = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{6!}$$

$$\approx 2.7181 \quad \text{(using calculator)}$$

One can also write a Matlab program to compute this sum.

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in a script file (.m)
    s = 1;
    for k = 1:6
        s = s + 1/factorial(k);
    end
    s
```
2. How large should $n$ be so that the function $f(x) = e^{-2x}$ can be approximated by its $n$’th Taylor polynomial $p_n(x)$ within error tolerance $\epsilon = 10^{-6}$ for all $x \in (-2, 1)$?

We already know that

$$g(t) = e^t = 1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots + \frac{t^n}{n!} + R_n(t).$$

Replace $t$ by $-2x$:

$$\frac{e^{-2x}}{f(x)} = 1 + \frac{-2x}{1!} + \frac{(-2x)^2}{2!} + \frac{(-2x)^3}{3!} + \cdots + \frac{(-2x)^n}{n!} + \frac{R_n(-2x)}{n!}$$

$q_n(x)$ (the $n$th Taylor poly. of $f$)  $\frac{R_n(-2x)}{n!}$ (the remainder)

We want to find $n$ such that $|R_n(x)| = |R_n(-2x)| < 10^{-6}$.

Let $t = -2x$.

Apply Lagrange’s theorem for function $g$:

$$R_n(t) = \frac{g^{(n+1)}(c)}{(n+1)!} t^{n+1} = \frac{e^c}{(n+1)!} t^{n+1}$$

where $c$ is some number between $0$ and $t = -2x$.

When $x$ varies on $(-2, 1)$, $t = -2x$ varies on $(-2, 4)$.

We see that $c$ must lie between $-2$ and $4$. Thus,

$$|R_n(-2x)| = \frac{e^c}{(n+1)!} (-2x)^{n+1} \leq \frac{e^4}{(n+1)!} \frac{2^{n+1}}{(n+1)!} < \frac{2^4 4^{n+1}}{(n+1)!} = \frac{8 4^n}{(n+1)!}$$
Therefore, the error term $R_n(x)$ is bounded by
\[ |R_n(x)| < \frac{81.4^{n+1}}{(n+1)!} \quad \forall x \in (-2, 1). \]

To make sure that $|R_n(x)| < 10^{-6}$ for all $x \in (-2, 1)$, we only need to choose $n$ such that
\[ \frac{81.4^{n+1}}{(n+1)!} < 10^{-6}. \]

$n = 20$ will do it (checking by calculator).