1. Let us compute approximately the integral \( I = \int_{\frac{1}{3}}^{\frac{1}{5}} \frac{1}{x^2 - 1} \, dx \) by midpoint rule (call the sum \( M_n \)) with equally spaced sample points \( 3 = x_0 < \ldots < x_n = 5 \).

(a) Write \( M_n \) using sigma notation.

(b) Find \( n \) such that \( M_n \) approximates \( I \) with error not exceeding \( \epsilon = 10^{-4} \).

\[
\begin{align*}
\frac{\delta}{n} &= \frac{\frac{5-3}{n}}{2} = \frac{2}{n}.
\end{align*}
\]

\[
\begin{align*}
x_0 &= 3, \ x_1 = 3 + \frac{2}{n}, \ x_2 = 3 + \frac{4}{n}, \ldots, \\
x_k &= 3 + \frac{2k}{n}.
\end{align*}
\]

The midpoint of the interval \([x_k, x_{k+1}]\) is

\[
\begin{align*}
x^*_k &= \frac{x_k + x_{k+1}}{2} = \frac{1}{2} \left( 3 + \frac{2k}{n} + 3 + \frac{2(k+1)}{n} \right) \\
&= 3 + \frac{2k+1}{n}.
\end{align*}
\]

The midpoint Riemann sum is

\[
\begin{align*}
M_n &= \frac{\delta}{n} f(x^*_0) + \frac{\delta}{n} f(x^*_1) + \ldots + \frac{\delta}{n} f(x^*_{n-1}) \\
&= \frac{2}{n} \sum_{k=0}^{n-1} f \left( 3 + \frac{2k+1}{n} \right) \\
&= \frac{2}{n} \sum_{k=0}^{n-1} \frac{1}{\left( 3 + \frac{2k+1}{n} \right)^2 - 1} \\
&= \frac{2}{n} \sum_{k=0}^{n-1} \frac{1}{\left( \frac{5n+2+2k}{n} \right)^2 - 1}.
\end{align*}
\]

We have

\[
\left| M_n - I \right| \leq \frac{(b-a)^3}{24 \delta^2} \max_{x \in [3,5]} |f'''(x)| = \frac{1}{3n^2} \max_{x \in [3,5]} |f'''(x)| \quad (\star)
\]

\[
\begin{align*}
\frac{(5-3)^3}{24 \delta^2} &= \frac{1}{5n^2}.
\end{align*}
\]
We have
\[ f(x) = \frac{1}{x^2 - 1} \]
\[ f'(x) = \frac{-2x}{(x^2 - 1)^2} \]
\[ f''(x) = \frac{-2(x^2 - 1)^2 - (-2x)2(2x)(x^2 - 1)}{(x^2 - 1)^4} \]
\[ = \frac{-2(x^2 - 1) + 8x^2}{(x^2 - 1)^4} \]
\[ = \frac{6x^2 + 2}{(x^2 - 1)^4} \]

For \( x \in [3, 5] \),
\[ |f''(x)| = \frac{6x^2 + 2}{(x^2 - 1)^4} \leq \frac{6(5)^2 + 2}{(3^2 - 1)^4} = \frac{19}{512}. \]

Therefore,
\[ \max_{3 \leq x \leq 5} |f''(x)| \leq \frac{19}{512} \]

From (x), we have
\[ |M_n - I| \leq \frac{1}{3n^2} \times \frac{19}{512}. \]

To make sure that \( |M_n - I| < 10^{-9} \), we only need to choose \( n \) such that
\[ \frac{1}{3n^2} \times \frac{19}{512} < 10^{-9}. \]

Any \( n > 12 \) would do it.
2. Approximate the integral in Problem 1 using Simpson’s rule with \( n = 4 \). How large should \( n \) be such that the Simpson sum \( S_n \) approximates \( I \) with error not exceeding \( \epsilon = 10^{-4} \)?

We will skip Simpson's rule.