The Prism Operator

1 Singular Homology

The *standard n-simplex* $\Delta^n \subset \mathbb{R}^{n+1}$ is the set

$$\Delta^n = \{(t_o, t_1, \ldots, t_n\,) \subset \mathbb{R}^{n+1} |\, \sum_{i=0}^{n} t_i = 1, \quad t_i \geq 0, \quad 0 \leq i \leq n\}.$$

A *singular n-simplex* in a topological space X is a map $\sigma^n \colon \Delta^n \to X$. Denote the free abelian group generated by singular simplices by $C_n(X)$ – thus elements σ of $C_n(X)$, known as *singular n-chains*, are formal finite sums $\sigma = \sum_i n_i \sigma_i$, $n_i \in \mathbb{Z}$.

Let $F_i^n \colon \Delta^{n-1} \to \Delta^n$ be the map

$$F_i^n(t_o, \ldots, t_{i-1}, t_i \ldots, t_{n-1}) = (t_o, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1}). \tag{1}$$

The *boundary map* $\partial_n = \partial \colon C_n(X) \to C_{n-1}(X)$ is the homomorphism defined on singular simplices by

$$\partial_n \sigma^n = \sum_{i=0}^{n} (-1)^i \sigma^n \circ F_i^n, \qquad n \geq 0.$$

We also let $\partial_0 = 0$. When there is no danger of confusion, we often omit the reference to dimension and simply write σ, ∂, etc.

Proposition 1. *The boundary maps satisfy $\partial_{n-1}\partial_n = 0$, $n \geq 1$.*

Proof. Let $j < i$. We compute

$$\begin{aligned} F_i^n \circ F_j^{n-1}(t_o, \ldots, t_{n-2}) &= (t_o, \ldots, t_{j-1}, 0, t_j, \ldots, t_{i-2}, 0, t_{i-1}, \ldots, t_{n-2}) \\ &= F_j^n \circ F_{i-1}^{n-1}(t_o, \ldots, t_{n-2}). \end{aligned}$$

Hence

$$\begin{aligned} \partial_{n-1}(\partial_n \sigma) &= \sum_{i=0}^{n} \sum_{j=0}^{n-1} (-1)^{i+j}\, \sigma \circ F_i^n \circ F_j^{n-1} \\ &= \sum_{j=0}^{n-1} \sum_{i=0}^{j} (-1)^{i+j}\, \sigma \circ F_i^n \circ F_j^{n-1} + \sum_{j=0}^{n-1} \sum_{i=j+1}^{n} (-1)^{i+j}\, \sigma \circ F_i^n \circ F_j^{n-1} \qquad (2) \\ &= \sum_{j=0}^{n-1} \sum_{i=0}^{j} (-1)^{i+j}\, \sigma \circ F_i^n \circ F_j^{n-1} + \sum_{j=0}^{n-1} \sum_{i=j+1}^{n} (-1)^{i+j}\, \sigma \circ F_j^n \circ F_{i-1}^{n-1}. \end{aligned}$$

But

$$\begin{aligned}\sum_{j=0}^{n-1}\sum_{i=j+1}^{n}(-1)^{i+j}\,\sigma\circ F_j^n\circ F_{i-1}^{n-1} &= \sum_{j=0}^{n-1}\sum_{i=j}^{n-1}(-1)^{i+j+1}\,\sigma\circ F_j^n\circ F_i^{n-1}\\ &= \sum_{i=0}^{n-1}\sum_{j=i}^{n-1}(-1)^{i+j+1}\,\sigma\circ F_i^n\circ F_j^{n-1} = \sum_{j=0}^{n-1}\sum_{i=0}^{j}(-1)^{i+j+1}\,\sigma\circ F_j^n\circ F_i^{n-1}.\end{aligned}\tag{3}$$

Now equations (2) and (3) together prove the claim. $\boxed{QED}$

As a corollary, the groups $C_*(X)$ alongside the boundary maps ∂ form a chain complex

$$\cdots\xrightarrow{\partial_{n+2}} C_{n+1}(X)\xrightarrow{\partial_{n+1}} C_n(X)\xrightarrow{\partial_n} C_{n-1}(X)\xrightarrow{\partial_{n-1}}\cdots,$$

called the *singular chain complex on X*. Singular chains belonging to $Z_n(X)=\ker\partial_n$ are called *n-cycles* while those in $B_n(X)=\operatorname{Im}\partial_{n+1}$ are *n-boundaries*. Obviously, $B_n(X)\subset Z_n(X)$, so that the *singular homology groups*

$$\mathrm{H_n(X)=Z_n(X)/B_n(X)},\qquad \mathrm{n}\geq 0,\tag{4}$$

of X are well defined.

A map $f\colon X\to Y$ induces a map on singular simplices by composition,

$$f_\sharp(\sigma^n)=f\circ\sigma^n,\qquad \text{where } \sigma^n\colon\Delta^n\to X.$$

We extend $f_\sharp$ linearly to maps $f_\sharp\colon C_n(X)\to C_n(Y)$, $n\geq 0$.

Proposition 2. *The maps $f_\sharp\colon C_n(X)\to C_n(Y)$ are chain maps; $f_\sharp\circ\partial_n=\partial_n\circ f_\sharp$, $n\geq 0$.*

Proof. By linearity it suffices to show that

$$f_\sharp(\partial_n\sigma)=\partial_n(f_\sharp\sigma)$$

for a singular simplex σ on X. We compute

$$\begin{aligned}f_\sharp(\partial_n\sigma) &= \sum_{i=0}^{n}(-1)^i f_\sharp(\sigma\circ F_i^n) = \sum_{i=0}^{n}(-1)^i f\circ(\sigma\circ F_i^n)\\ &= \sum_{i=0}^{n}(-1)^i(f\circ\sigma)\circ F_i^n = \partial_n(f\circ\sigma)=\partial_n(f_\sharp\sigma).\end{aligned}$$

$\boxed{QED}$

Thus by Proposition 2, the maps $f_\sharp$ descend to maps

$$f_\star\colon \mathrm{H_n(X)\to H_n(Y)},\qquad \mathrm{n}\geq 0,$$

in homology by the prescription $f_\star[\sigma]=[f_\sharp\sigma]$, where $\sigma\in Z_n(X)$. As is easily verified, $f_\star$ yields a well-defined map between homology classes.

2 Homotopies

Maps f, $g\colon X \to Y$ are homotopic provided there is a homotopy connecting f to g, that is, a map $H\colon X \times I \to Y$ with $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$. Here $I = [0,1]$ denotes the closed unit interval.

Our goal is to show that homotopic maps induce identical maps in homology. For this we start by defining maps $E_i^n\colon \Delta^{n+1} \to \Delta^n$ and $S_i\colon \Delta^{n+1} \to I$, $0 \le i \le n$, by

$$\begin{aligned} E_i^n(t_o, t_1, \ldots, t_{n+1}) &= (t_o, \ldots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \ldots, t_{n+1}), && (5)\\ S_i(t_o, t_1, \ldots, t_{n+1}) &= t_{i+1} + \cdots + t_{n+1}. && (6)\end{aligned}$$

Proposition 3. *Let* F_i^n, E_i^n, S_i *be as in equations* (1), (5), *and* (6). *Then*

$$E_i^n \circ F_j^{n+1} = \begin{cases} F_{j-1}^n \circ E_i^{n-1}, & i \le j-2; \\ F_j^n \circ E_{i-1}^{n-1}, & i \ge j+1; \end{cases} \tag{7}$$

$$E_i^n \circ F_i^{n+1} = E_i^n \circ F_{i+1}^{n+1} = id; \tag{8}$$

$$S_i \circ F_j^{n+1} = \begin{cases} S_i, & \text{if } i < j; \\ S_{i-1}, & \text{if } i \ge j. \end{cases} \tag{9}$$

Proof. The proof is based on direct computations. Assuming that $i \le j-2$,we have

$$\begin{aligned} E_i^n \circ F_j^{n+1}(t_o, \ldots, t_n) &= E_i^n(t_o, \ldots, t_{j-1}, 0, t_j, \ldots, t_n) \\ &= (t_o, \ldots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \ldots, t_{j-1}, 0, t_j, \ldots, t_n) \\ &= F_{j-1}^n(t_o, \ldots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \ldots, t_{j-1}, t_j, \ldots, t_n) \\ &= F_{j-1}^n \circ E_i^{n-1}(t_o, \ldots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \ldots, t_{j-1}, t_j, \ldots, t_n), \end{aligned}$$

which proves the first equation in (7).

Next assume that $i \ge j+1$. We compute

$$\begin{aligned} E_i^n \circ F_j^{n+1}(t_o, \ldots, t_n) &= E_i^n(t_o, \ldots, t_{j-1}, 0, t_j, \ldots, t_n) \\ &= (t_o, \ldots, t_{j-1}, 0, t_j, \ldots, t_{i-2}, t_{i-1} + t_i, t_{i+1}, \ldots, t_n) \\ &= F_j^n(t_o, \ldots, t_{j-1}, t_j, \ldots, t_{i-2}, t_{i-1} + t_i, t_{i+1}, \ldots, t_n) \\ &= F_j^n \circ E_{i-1}^{n-1}(t_o, \ldots, t_{j-1}, t_j, \ldots, t_{i-2}, t_{i-1}, t_i, t_{i+1}, \ldots, t_n). \end{aligned}$$

Thus the second equation in (7) holds true.

Now let $j = i$ or $j = i+1$. In each instance we have that

$$\begin{aligned} E_i^n \circ F_j^{n+1}(t_o, \ldots, t_n) &= E_i^n(t_o, \ldots, t_{j-1}, 0, t_j, \ldots, t_n) \\ &= (t_o, \ldots, t_{j-1}, t_j, \ldots, t_n), \end{aligned}$$

which proves (8).

Finally, we compute

$$S_i \circ F_j^{n+1}(t_o, \ldots, t_n) = S_i(t_o, \ldots, t_{j-1}, 0, t_j, \ldots, t_n) = \begin{cases} t_{i+1} + \cdots + t_n, & \text{if } i < j; \\ t_i + \cdots + t_n, & \text{if } i \geq j. \end{cases} \tag{10}$$

This concludes the proof of the Proposition.

$\boxed{QED}$

Now given a map $H\colon X \times I \to Y$, the *prism operator* $\mathrm{P}_H\colon C_n(X) \to C_{n+1}(Y)$, $n \geq 0$, associated with H is the homomorphisms defined on singular simplices by

$$\mathrm{P}_H(\sigma^n) = \sum_{i=0}^{n} (-1)^i H(\sigma^n \circ E_i^n, S_i), \tag{11}$$

that is,

$$\mathrm{P}_H(\sigma^n)(t_o, \ldots, t_{n+1}) = \sum_{i=0}^{n} (-1)^i H(\sigma^n(t_o, \ldots, t_{i-1}, t_i + t_{i+1}, \ldots, t_{n+1}), t_{i+1} + \cdots + t_{n+1}).$$

We compute

$$\begin{aligned} \partial \mathrm{P}_H(\sigma^n) &= \sum_{i=0}^{n} (-1)^i \partial\big(H(\sigma^n \circ E_i^n, S_i)\big) \\ &= \sum_{i=0}^{n} \sum_{j=0}^{n+1} (-1)^{i+j} H(\sigma^n \circ E_i^n \circ F_j^{n+1}, S_i \circ F_j^{n+1}). \end{aligned} \tag{12}$$

By proposition 3,

$$\sigma^n \circ E_i^n \circ F_{i+1}^{n+1} = \sigma^n \circ E_{i+1}^n \circ F_{i+1}^{n+1}, \qquad S_i \circ F_{i+1}^{n+1} = S_{i+1} \circ F_{i+1}^{n+1}.$$

Hence the terms in (12) corresponding to the index values $i = k$, $j = k+1$ and $i = k+1$,

$j = k+1$, $0 \leq k \leq n-1$, cancel pairwise. Thus (12) reduces to

$$
\begin{aligned}
\partial \mathrm{P}_H(\sigma^n) &= H(\sigma^n,1) - H(\sigma^n,0) + \sum_{i=1}^{n}\sum_{j=0}^{i-1}(-1)^{i+j} H(\sigma^n \circ E_i^n \circ F_j^{n+1}, S_i \circ F_j^{n+1}) \\
&\qquad + \sum_{i=0}^{n-1}\sum_{j=i+2}^{n+1}(-1)^{i+j} H(\sigma^n \circ E_i^n \circ F_j^{n+1}, S_i \circ F_j^{n+1}), \\
&= H(\sigma^n,1) - H(\sigma^n,0) + \sum_{i=1}^{n}\sum_{j=0}^{i-1}(-1)^{i+j} H(\sigma^n \circ F_j^n \circ E_{i-1}^{n-1}, S_{i-1}) \\
&\qquad + \sum_{i=0}^{n-1}\sum_{j=i+2}^{n+1}(-1)^{i+j} H(\sigma^n \circ F_{j-1}^n \circ E_i^{n-1}, S_i), \\
&= H(\sigma^n,1) - H(\sigma^n,0) + \sum_{i=0}^{n-1}\sum_{j=0}^{i}(-1)^{i+j+1} H(\sigma^n \circ F_j^n \circ E_i^{n-1}, S_i) \\
&\qquad + \sum_{i=0}^{n-1}\sum_{j=i+1}^{n}(-1)^{i+j+1} H(\sigma^n \circ F_j^n \circ E_i^{n-1}, S_i) \\
&= H(\sigma^n,1) - H(\sigma^n,0) - \sum_{i=0}^{n-1}\sum_{j=0}^{n}(-1)^{i+j} H(\sigma^n \circ F_j^n \circ E_i^{n-1}, S_i) \\
&= H(\sigma^n,1) - H(\sigma^n,0) - \mathrm{P}_H\Big(\sum_{j=0}^{n}(-1)^j \sigma^n \circ F_j^n\Big) \\
&= H(\sigma^n,1) - H(\sigma^n,0) - \mathrm{P}_H\big(\partial\sigma^n\big),
\end{aligned}
\tag{13}
$$

where in the second step we used the equations of Proposition 3. In the above expressions the first and second terms on the right-hand side arise from the summands in (12) corresponding to the index values $i = j = 0$ and $i = n$, $j = n+1$, respectively.

Lemma 4. *Let $H\colon X \times I \to Y$ be a homotopy between maps $f\colon X \to Y$ and $g\colon X \to Y$, and let P_H be the prism operator associated with H. Then*

$$
g_\sharp(\sigma^n) - f_\sharp(\sigma^n) = \partial(P_H\sigma^n) + P_H(\partial\sigma^n), \qquad \textit{for all } \sigma^n \in C_n(X),\ n \geq 0. \tag{14}
$$

Proof. Now $H(\cdot,0) = f$, $H(\cdot,1) = g$, so the result immediately follows from the computation in (13). $\boxed{\mathit{QED}}$

In general, a *chain complex* is composed of a sequence $\mathbf{C} = \{C_n, \partial_n\}$ of abelian groups C_n, $n \in \mathbb{Z}$, and homomorphisms $\partial_n\colon C_n \to C_{n-1}$ such that $\partial_n \circ \partial_{n-1} = 0$. Write

$\ker \partial_n = Z_n(\mathbf{C})$, $\operatorname{Im} \partial_{n+1} = B_n(\mathbf{C})$. Elements of $Z_n(\mathbf{C})$ and $B_n(\mathbf{C})$ are known as n-cycles and n-boundaries, respectively. The quotient group

$$\mathrm{H_n}(\mathbf{C}) = \frac{\mathrm{Z_n}(\mathbf{C})}{\mathrm{B_n}(\mathbf{C})}$$

is called the n^{th} homology group of $\mathbf{C}$.

A *chain map* $\mathbf{f} = \{f_n\}$ between two chain complexes $\mathbf{C} = \{C_n, \partial_n\}$, $\mathbf{D} = \{D_n, \partial'_n\}$ consists of homomorphisms $f_n \colon C_n \to D_n$ with $\partial'_n f_n = f_{n-1}\partial_n$. Clearly a chain map sends n-cycles/n-boundaries in $\mathbf{C}$ to n-cycles/n-boundaries in $\mathbf{D}$. Consequently, a chain map $\mathbf{f} = \{f_n\}$ induces well-defined maps

$$f_{n\star} \colon \mathrm{H_n}(\mathbf{C}) \to \mathrm{H_n}(\mathbf{D})$$

in homology by $f_{n\star}[\sigma] = [f_n\sigma]$, where the square brackets indicate the class of an n-cycle in homology.

Two chain maps $\mathbf{f} = \{f_n\}$, $\mathbf{g} = \{g_n\}$ are *chain homotopic* provided that there are homomorphisms $h_n : C_n \to D_{n+1}$ so that

$$f_n - g_n = h_{n-1}\partial_n \pm \partial'_{n+1}h_n. \tag{15}$$

Given a n-cycle σ in $\mathbf{C}$, definition (15) implies that

$$f_n\sigma = g_n\sigma \pm \partial'_{n+1}(h_n\sigma),$$

so that the difference between $f_n\sigma$ and $g_n\sigma$ is a boundary in $\mathbf{D}$. It follows that $\mathbf{f}$ and $\mathbf{g}$ induce identical mappings in homology, $f_{n\star} = g_{n\star}$ for all n.

Theorem 5. *Let $H \colon X \times I \to Y$ be a homotopy between maps $f \colon X \to Y$ and $g \colon X \to Y$, and let P_H be the prism operator associated with H. Then P_H provides a chain homotopy between the chain maps $f_\sharp, g_\sharp \colon C_*(X) \to C_*(Y)$. Thus f and g induce identical maps in singular homology, $f_\star = g_\star \colon \mathrm{H}_*(\mathrm{X}) \to \mathrm{H}_*(\mathrm{Y})$.*

The proof of Theorem 5 is a direct consequence of Lemma 4.

Recall that spaces X, Y are homotopy equivalent if there are maps $f \colon X \to Y$, $g \colon Y \to X$ so that the compositions $g \circ f$ and $f \circ g$ are homotopic to the identity maps on X and Y, respectively.

Corollary 6. *Suppose X and Y are homotopy equivalent under $f \colon X \to Y$. Then $f_\star \colon \mathrm{H}_*(\mathrm{X}) \to \mathrm{H}_*(\mathrm{Y})$ is an isomorphism.*