Variational Principles for Natural Divergence-free Tensors in Metric Field Theories

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Abstract

Let $\mathrm{T}^{ab} = \mathrm{T}^{ba} = 0$ be a system of differential equations for the components of a metric tensor on $\mathbb{R}^m$. Suppose that T^{ab} transforms tensorially under the action of the diffeomorphism group on metrics and that the covariant divergence of T^{ab} vanishes. We then prove that $\mathrm{T}^{ab} = \mathrm{E}^{ab}(L)$ is the Euler-Lagrange expression some Lagrangian density L provided that T^{ab} is of third order. Our result extends the classical works of Cartan, Weyl, Vermeil, Lovelock, and Takens on identifying field equations for the metric tensor with the symmetries and conservation laws of the Einstein equations.

Keywords: metric field theories, symmetries of differential equations, conservation laws, inverse problem of the calculus of variations.

MSC 2010 classification. Primary: 58E30, 58J70, Secondary: 35Q75.

1 Introduction and main results

The time-honored Noether's theorems [23] establish a correspondence between symmetries and conservation laws for the Euler-Lagrange equations of a classical variational problem. Noether's first theorem states that every infinitesimal symmetry of the variational problem determines a differential conservation law for the Euler-Lagrange equations, and conversely that, under certain mild non-degeneracy conditions, conservation laws can be associated with symmetries of the variational problem. Noether's second theorem, in turn, asserts that infinite dimensional symmetry pseudogroups of the variational problem involving arbitrary functions of all the independent variables correspond to differential identities among the equations.

In 1977, F. Takens [27] considered the following novel and very distinct aspect of the interplay between symmetries, conservation laws and variational principles for systems of differential equations:

> *Let Γ be a Lie algebra of vector fields defined on the space of independent and dependent variables, and suppose that a system of differential equations is invariant under Γ and that each element in Γ generates a conservation law for the system. Does it then follow that the system arises from a variational principle?*

In his original paper Takens studied the question for second order scalar equations, systems of linear equations, and equations arising in metric field theories. Takens' results for second order scalar equations and for systems of linear equations were subsequently generalized by Anderson and Pohjanpelto [7], [8], [24]. We refer to [7] in particular for more background material and motivation on Takens' problem.

In addition to the papers listed above, there is an extensive literature dealing with the existence of variational principles for systems of differential equations admitting a Lie algebra of symmetries and the corresponding conservation laws within the context of Noether's second theorem, that is, where the symmetry group is the infinite dimensional group of coordinate transformations of the underlying manifold and the conservation laws express the vanishing of the covariant divergence (or some variant of it) of the field equations. This work is largely directed towards the axiomatic characterization of the Einstein equations. The original classification results of Cartan [11], Vermeil [28], and Weyl [29] establish that second order quasi-linear field equations for the metric tensor possessing the symmetries and conservation laws of the Einstein equations necessarily arise from a variational principle. These results were later extended to general, fully nonlinear second order systems for the metric tensor and to third order systems in the 3-dimensional case by Lovelock, [16, 17, 19], again by a detailed classification of all equations with the required properties. A direct proof for second order systems based on the analysis of the Helmholtz conditions [2], i.e., the integrability conditions for the existence of a variational principle, can be found in [27]. Lovelock's results were later generalized to metric-scalar [13], [14], metric-vector [18], and metric-bivector [20] theories. Anderson [3] subsequently provides a general characterization of second order divergence free systems for the metric tensor and an auxiliary independent tensor field, subsuming in part the above-mentioned works on combined metric field theories.

In [15], Horndeski attempts to extend Lovelock's work to general third order equations with the symmetries and conservation laws of the Einstein equations. However, the treatment for the existence of a variational principle falls short of a comprehensive result due to a restrictive extraneous assumption that the zeroth order Helmoltz conditions for the system be invariant under the action of the diffeomorphism group.

The next important step extending Lovelock's work was the introduction of the generalized Cotton tensors in [1]. These tensors and their construction is placed into the proper differential geometric context by Anderson [5] as the Euler-Lagrange expressions of Lagrangians derived from the Chern-Simons forms of Riemannian geometry. As is discovered in [5], these analogs of the classical Cotton tensors of 3-dimensional conformal geometry play a key role in the equivariant inverse problem of the calculus of variations for metrics.

The aim of this paper is to extend the results of Cartan, Vermeil, Weyl, and Lovelock to general third order field equations for the metric tensor with the symmetries and conservation laws of the Einstein equations. As is intimated by the intricacy of the constructions in [1], [5], a direct classification of third order equations along the lines of the original works on the subject would be a formidable undertaking. However, as is well known, the existence of a local Lagrangian for a system of differential equations is ensured by the vanishing of the classical Helmholtz conditions for the equations, and our problem is rendered tractable by an analysis of these conditions for systems sharing the properties of the Einstein equations.

To describe our results more precisely, write (x^a) for the coordinates on $\mathbb{R}^m$. A metric

$$\mathsf{g} = g_{ab} dx^a \otimes dx^b$$

on $\mathbb{R}^m$ is a symmetric type $(0,2)$ tensor field with $g = \det(g_{ab}) \neq 0$, where the g_{ab} stand for the components of g. In this paper we consider metrics of fixed but arbitrary signature. The action of the Lie algebra $\mathcal{X}(\mathbb{R}^m)$ of vector fields on metric tensors via pull-back gives rise to the infinitesimal transformation group

$$\mathfrak{g} = \left\{ X_\xi = \xi^i \frac{\partial}{\partial x^i} - 2\xi^c_{,(a} g_{b)c} \frac{\partial}{\partial g_{ab}} \mid \xi^a \in \mathcal{X}(\mathbb{R}^m) \right\} \tag{1.1}$$

on the coordinate space $\mathsf{G} = \{(x^i, g_{bc})\}$.

The metric tensor g is subject to a system of kth order partial differential equations

$$\mathrm{T}^{ab} = \mathrm{T}^{(ab)}(x^i, g_{cd}, g_{cd,i_1}, g_{cd,i_1 i_2}, \dots, g_{cd,i_1 i_2 \cdots i_k}) = 0, \qquad a, b = 1, \dots, m,$$

where $g_{cd,i_1 i_2 \cdots i_l}$ denotes the derivative of g_{cd} with respect to the independent variables x^{i_1},x^{i_2},..., x^{i_l}. The operator T^{ab} is locally variational if it can be written in some neighborhood of each point of its domain as the Euler–Lagrange expression

$$\mathrm{T}^{ab} = \mathrm{E}^{ab}(L) = \frac{\partial L}{\partial g_{ab}} - D_{i_1}\left(\frac{\partial L}{\partial g_{ab,i_1}}\right) + D_{i_1} D_{i_2}\left(\frac{\partial L}{\partial g_{ab,i_1 i_2}}\right) - \cdots \tag{1.2}$$

of some locally defined Lagrangian

$$L = L(x^c, g_{cd}, g_{cd,i_1}, g_{cd,i_1 i_2}, \dots, g_{cd,i_1 i_2 \cdots i_l})$$

depending on the components of the metric tensor and their derivatives. Here D_i denotes the standard coordinate total derivative operator.

If the Lagrangian L transforms as a scalar density, that is,

$$\mathcal{L}_{\mathrm{pr}\,X_\xi} L = \mathrm{pr}\,X_\xi(L) + \mathrm{div}\,\xi\, L = 0, \qquad \text{for all } X_\xi \in \mathfrak{g}, \tag{1.3}$$

or, equivalently, the density $L_g = g^{-1/2}L$ is invariant under the prolonged action of $\mathfrak{g}$, then the Euler-Lagrange expressions $\mathrm{T}^{ab} = \mathrm{E}^{ab}(L)$ constitute a tensor density, whereby

$$\mathcal{L}_{\mathrm{pr}\,X_\xi} \mathrm{T}^{ab} = \mathrm{pr}\,X_\xi(\mathrm{T}^{ab}) - 2\xi^{(a}_{,c}\mathrm{T}^{b)c} + \mathrm{div}\,\xi\, \mathrm{T}^{ab} = 0, \qquad \text{for all } X_\xi \in \mathfrak{g}. \tag{1.4}$$

Here $\mathcal{L}$ denotes the standard Lie derivative operator. Lagrangians L and differential operators T^{ab} satisfying (1.3) or (1.4) are also known as *natural* tensor densities.

In addition, in light of the diffeomorphism invariance (1.4), Noether's second theorem implies that the components $\mathrm{T}^{ab} = \mathrm{E}^{ab}(L)$ are divergence-free,

$$D_b\mathrm{T}^{ab} + \Gamma^a_{bc}\mathrm{T}^{bc} = 0, \tag{1.5}$$

where the Γ^a_{bc} denote the standard Christoffel symbols of the metric g.

In the present paper we investigate a partial converse to Noether's second theorem for third order operators for metrics, that is, whether a symmetric, type $(2,0)$ differential operator $\mathrm{T}^{ab} = \mathrm{T}^{(ab)}(x^i, g_{cd}, g_{cd,i_1}, g_{cd,i_1i_2}, g_{cd,i_1i_2i_3})$ satisfying the invariance condition (1.4) and subject to the differential constraints (1.5) coincides with the Euler-Lagrange expression of some Lagrangian L.

Our main result is the following.

Theorem 1. *Suppose that a third order differential operator*

$$\mathrm{T}^{ab} = \mathrm{T}^{(ab)}(x^i, g_{cd}, g_{cd,i_1}, g_{cd,i_1i_2}, g_{cd,i_1i_2i_3}), \qquad a,b = 1,\dots,m,$$

admits the symmetries (1.4) *and satisfies the differential constraints* (1.5)*. Then* T^{ab} *is locally variational. Moreover, suppose that* T^{ab} *is everywhere smooth. Then it can be written as*

$$\mathrm{T}^{ab} = \begin{cases} \mathrm{E}^{ab}(L), & \text{if } m \equiv 0,1,2 \mod 4, \\ \mathrm{E}^{ab}(L) + C^{ab}_{\mathsf{P}}, & \text{if } m \equiv 3 \mod 4, \end{cases} \tag{1.6}$$

where the Lagrangian L *is a scalar density satisfying* (1.3) *and* C^{ab}_{P} *is the generalized Cotton tensor associated with an invariant polynomial* P *on* $\mathfrak{so}(p,q)$*,* $p+q=m$*, of degree* $(m+1)/2$*.*

The generalized Cotton tensors C^{ab}_{P} are, therefore, locally variational, but, as is proved in [5], they can not be written as the Euler-Lagrange expressions of a natural Lagrangian. In the physically most relevant situation with $m=4$, Theorem 1 asserts that if a third order, natural system of differential equations

$$\mathrm{T}^{ab}(g_{cd}, g_{cd,i_1}, g_{cd,i_1i_2}, g_{cd,i_1i_2i_3}) = 0, \qquad \text{where } \mathrm{T}^{ab} = \mathrm{T}^{(ab)},$$

for the components g_{cd} of the metric tensor is divergence free (1.5), then there is a natural Lagrangian L so that $\mathrm{T}^{ab} = \mathrm{E}^{ab}(L)$.

This paper is organized as follows. After covering some preliminary material relevant to the problem at hand in section 2, we analyze in section 3 the relationship between symmetries (1.4) and the conservation law (1.5) for metric field equations. In particular, we show that any natural differential operator T^{ab} admitting translational conservation laws is necessarily divergence free. This interesting though elementary fact does not seem to have been previously noted in the literature. Then in section 4 we present the proof of Theorem 1 and, finally, in section 5 we discuss some open problems and generalizations of the work at hand.

2 Preliminaries

In this section we collect together some basic definitions and results from the formal calculus of variations on jet spaces germane to the problem at hand. For more details and proofs we refer, e.g., to [2, 23].

Let $\mathsf{G} \to \mathbb{R}^m$ be the trivial bundle of metrics, that is, of non-degenerate symmetric bilinear forms on $\mathbb{R}^m$ with fixed signature. Denote the coordinates of $\mathbb{R}^m$ by $(x^1, x^2, \dots, x^m)$. Then the components $g_{ab} = g_{(ab)}$ of a metric g are determined by $\mathsf{g} = g_{ab} dx^a \otimes dx^b$, so that, as a coordinate bundle,

$$\mathsf{G} = \{(x^i, g_{ab})\} \to \{(x^i)\}, \qquad \text{where } a \le b.$$

We denote the bundle of order k jets, $0 \le k \le \infty$, of local sections of G by $J^k(\mathsf{G})$; in the induced coordinates

$$J^k(\mathsf{G}) = \{(x^i, g_{ab}, g_{ab,i_1}, g_{ab,i_1 i_2}, \dots, g_{ab,i_1 i_2 \cdots i_l}, \dots, g_{ab,i_1 i_2 \cdots i_k})\}, \tag{2.1}$$

where $g_{ab,i_1 i_2 \cdots i_l}$, $a \le b$, stands for the l-th order derivative variables. For notational convenience we let $g_{ab,i_1 i_2 \cdots i_l} = g_{ba,i_1 i_2 \cdots i_l}$, when $a > b$. We also use $g^{[k]}$ to collectively designate all the variables $g_{ab,i_1 \cdots i_p}$, $p = 0, \dots, k$, up to order k.

Let $I = (i_1, i_2, \dots, i_l)$, $1 \le i_p \le m$, denote an unordered multi-index of length $|I| = l$. Define partial derivative operators $\partial^{ab,I}$, $1 \le a, b \le m$, $|I| \ge 0$, by

$$\partial^{ab,I} g_{cd,J} = \begin{cases} \delta_{(c}^{\ a} \delta_{d)}^{b} \delta_{\ j_1}^{(i_1} \delta_{j_2}^{i_2} \cdots \delta_{j_k}^{i_k)}, & \text{if } |I| = |J|, \\ 0, & \text{if } |I| \ne |J|, \end{cases} \tag{2.2}$$

where round brackets indicate symmetrization in the enclosed indices. Then, for example, the standard coordinate total derivative operators D_i on $J^\infty(\mathsf{G})$ are given in terms of the differential operators (2.2) by

$$D_i = \frac{\partial}{\partial x^i} + \sum_{|I| \ge 0} g_{ab,Ii} \partial^{ab,I}, \qquad i = 1, 2, \dots, m. \tag{2.3}$$

The expression (2.3) leads to the commutation formula

$$[\partial^{ab,I}, D_j] = \partial^{ab,(i_1\cdots i_{k-1}}\delta_j^{i_k)}. \tag{2.4}$$

We will employ the standard Einstein summation convention in what follows.

The flow of a vector field

$$X = P^i(x^j, g_{cd})\frac{\partial}{\partial x^i} + Q_{ab}(x^j, g_{cd})\partial^{ab} \tag{2.5}$$

on G induces a transformation on the space of sections of G, and, hence, by differentiation, it generates a local 1-parameter transformation group acting on $J^k(\mathsf{G})$, $k \geq 0$. The associated infinitesimal generator is called the k-th order *prolongation* of X and is denoted by $\mathrm{pr}^k X$. The components of $\mathrm{pr}^k X$ are given by the usual prolongation formula

$$\mathrm{pr}^k X = P^i D_i + \sum_{|I|\leq k} D_I(X_{\mathrm{ev},ab})\partial^{ab,I}, \tag{2.6}$$

where the $X_{\mathrm{ev},ab}$ denote the components of the *evolutionary form*

$$X_{\mathrm{ev}} = (Q_{ab} - P^c g_{ab,c})\partial^{ab}$$

of X and where, given a multi-index $I = (i_1, \dots, i_k)$, we use the abbreviated notation $D_I = D_{i_1}\cdots D_{i_k}$. We will also write $\mathrm{pr}^\infty X = \mathrm{pr}\, X$. The vector field (2.5) is called *projectable* if the coefficients $P^a = P^a(x^i)$ are functions of the independent variables x^i only. In particular, the infinitesimal generators of the action by the lifted diffeomorphism group (1.1) form a Lie algebra $\mathfrak{g}$ of projectable vector fields on G with

$$X_{\xi,\mathrm{ev},ab} = -2\xi^c_{,(a}g_{b)c} - \xi^c g_{ab,c}.$$

We associate to a given differential operator $\mathrm{T}^{ab} = \mathrm{T}^{ab}(x^i, g^{[k]})$ the *source form*

$$\mathrm{T} = \mathrm{T}^{ab} dg_{ab} \wedge \nu, \tag{2.7}$$

where $\nu = dx^1 \wedge dx^2 \wedge \cdots \wedge dx^m$ is the standard volume form on $\mathbb{R}^m$. The source form T is called *natural* if T is invariant under the prolonged action of the diffeomorphism group $\mathfrak{g}$, that is,

$$\mathcal{L}_{\mathrm{pr}\, X_\xi}\mathrm{T} = 0, \qquad \text{for all } \xi \in \mathcal{X}(\mathbb{R}^m),$$

where X_ξ is defined in (1.1). As is easy to verify, a source form $\mathrm{T} = \mathrm{T}^{ab}dg_{ab}\wedge\nu$ is natural precisely when the components T^{ab} form a tensor density as in (1.4). A vector field X on G *generates a conservation law* for T if there are differential functions $t^i = t^i(x^j, g^{[l]})$, $i = 1, \dots, m$, so that

$$X_{\mathrm{ev},ab}\mathrm{T}^{ab} = D_i t^i. \tag{2.8}$$

The source form T is said to be derivable from a *variational principle* if there is a *Lagrangian function* $L = L(x^i, g^{[l]})$ such that T is the *Euler-Lagrange expression* of L, i.e.,

$$\mathrm{T}^{ab} = \mathrm{E}^{ab}(L) = \sum_{|I|\geq 0}(-D)_I(\partial^{ab,I}L). \tag{2.9}$$

It will be convenient to call

$$\lambda = L(x^a, g^{[l]})\,\nu$$

a *Lagrangian n-form* and

$$\mathrm{E}(\lambda) = \mathrm{E}^{ab}(L)\,dg_{ab} \wedge \nu$$

the *Euler-Lagrange form* associated with λ. As is well known [2], the Euler-Lagrange operator commutes with the prolonged action of projectable transformations on G; infinitesimally,

$$\mathrm{E}(\mathcal{L}_{\mathrm{pr}\,X}\lambda) = \mathcal{L}_{\mathrm{pr}\,X}\mathrm{E}(\lambda), \tag{2.10}$$

for every projectable vector field X and Lagrangian form λ, where $\mathcal{L}$ denotes the standard Lie derivative operator.

The Helmholtz operator H_T acts on evolutionary vector fields $Y = Y_{ab}\partial^{ab}$ on G by

$$\mathrm{H}_\mathrm{T}(Y) = \mathcal{L}_{\mathrm{pr}\,Y}\mathrm{T} - \mathrm{E}(Y \lrcorner\, \mathrm{T}), \tag{2.11}$$

cf. [9]. If we write

$$\mathrm{H}_\mathrm{T}(Y) = \sum_{|I|\geq 0}(D_I Y_{cd})\mathrm{H}_\mathrm{T}^{ab,cd,I}dg_{ab} \wedge \nu, \tag{2.12}$$

then the components $\mathrm{H}_\mathrm{T}^{ab,cd,I}$ of H_T are explicitly given by

$$\mathrm{H}_\mathrm{T}^{ab,cd,I} = \partial^{cd,I}\mathrm{T}^{ab} - (-1)^{|I|}\mathrm{E}^{ab,I}(\mathrm{T}^{cd}), \qquad |I| \geq 0. \tag{2.13}$$

Here the $\mathrm{E}^{ab,I}$ denote the higher Euler-Lagrange operators [2] acting on a differential function F defined on some $J^k(\mathsf{G})$ by

$$\mathrm{E}^{ab,I}(F) = \sum_{|J|\geq 0}\binom{|I|+|J|}{|I|}(-D)_J(\partial^{ab,IJ}F), \qquad |I| \geq 0.$$

Note that if $\mathrm{T} = \mathrm{T}^{ab}(x^i, g^{[k]})dg_{ab} \wedge \nu$ is of order k, then $\mathrm{H}_\mathrm{T}^{ab,cd,I} = 0$ for $|I| > k$ and for $|I| = 0, \dots, k$, the components $\mathrm{H}_\mathrm{T}^{ab,cd,I}$ are of order at most $2k - |I|$.

It is not difficult to see that a source form $\mathrm{T} = \mathrm{E}(L)$ deriving from a variational principle satisfies the Helmholtz conditions $\mathrm{H}_\mathrm{T} \equiv 0$, or, in components, $\mathrm{H}_\mathrm{T}^{ab,cd,I} = 0$. Conversely, one can show [2] that if the Helmholtz conditions $\mathrm{H}_\mathrm{T} \equiv 0$ are satisfied, then, at least locally, that is, in some neighborhood of each point in its domain in $J^k(\mathsf{G})$, the source form T can be written as the Euler-Lagrange expression of some Lagrangian L. Accordingly, we will call a source form satisfying the Helmholtz conditions *locally variational.*

Proposition 2. *Suppose that $X = P^i\partial/\partial x^i + Q_{ab}\partial^{ab}$ is a projectable vector field on G and that a source form* T *is invariant under the prolongation* $\mathrm{pr}\,X$ *of X. Then the components $\mathrm{H}_\mathrm{T}^{ab,cd,I}$ of the Helmholtz operator H_T associated with* T *satisfy the invariance conditions*

$$\mathrm{pr}\,X(\mathrm{H}_\mathrm{T}^{ab,cd,I}) + \sum_{|J|\geq|I|}\mathrm{H}_\mathrm{T}^{ab,ef,J}\partial^{cd,I}Q_{ef,J} + \mathrm{H}_\mathrm{T}^{ef,cd,I}\partial^{ab}Q_{ef} + \frac{\partial P^j}{\partial x^j}\mathrm{H}_\mathrm{T}^{ab,cd,I} = 0, \tag{2.14}$$

where $Q_{cd,J}$ denotes the $g_{cd,J}$-component of $\mathrm{pr}\,X$.

Proof. We first compute

$$\begin{aligned}\mathcal{L}_{\mathrm{pr}\,X}\left(\mathrm{H}_{\mathrm{T}}(Y)\right) &= \mathcal{L}_{\mathrm{pr}\,X}(\mathcal{L}_{\mathrm{pr}\,Y}\mathrm{T}) - \mathcal{L}_{\mathrm{pr}\,X}\mathrm{E}(Y \lrcorner\, \mathrm{T}) \\ &= \mathcal{L}_{\mathrm{pr}[X,Y]}\mathrm{T} - \mathrm{E}((\mathcal{L}_{\mathrm{pr}\,X}Y) \lrcorner\, \mathrm{T}) = \mathrm{H}_{\mathrm{T}}([X,Y]),\end{aligned} \tag{2.15}$$

where we used the invariance of T and the equivariance of the Euler-Lagrange operator under the prolonged action of projectable transformations. Next write $Y = Y_{cd}\partial^{cd}$. Then, on account of (2.12),

$$\begin{aligned}\mathcal{L}_{\mathrm{pr}\,X}\left(\mathrm{H}_{\mathrm{T}}(Y)\right) = \sum_{|I|\geq 0}\Bigg(&\mathrm{pr}\,X(\mathrm{H}_{\mathrm{T}}^{ab,cd,I})D_I Y_{cd} + \mathrm{H}_{\mathrm{T}}^{ab,cd,I}\,\mathrm{pr}\,X(D_I Y_{cd}) \\ &+ \mathrm{H}_{\mathrm{T}}^{ef,cd,I}(D_I Y_{cd})\partial^{ab}Q_{ef} + \frac{\partial P^j}{\partial x^j}\,\mathrm{H}_{\mathrm{T}}^{ab,cd,I}D_I Y_{cd}\Bigg)dg_{ab}\wedge\nu.\end{aligned} \tag{2.16}$$

The identity

$$[\mathrm{pr}\,X, \mathrm{pr}\,Y] = \mathrm{pr}[X,Y]$$

yields

$$\begin{aligned}\mathrm{pr}\,X(D_I Y_{cd}) &= D_I([X,Y]_{cd}) + \mathrm{pr}\,Y(Q_{cd,I}) \\ &= D_I([X,Y]_{cd}) + \sum_{|J|\geq 0}(D_J Y_{ef})\partial^{ef,J}Q_{cd,I}.\end{aligned} \tag{2.17}$$

Now by virtue of (2.17), equation (2.16) becomes

$$\begin{aligned}\mathcal{L}_{\mathrm{pr}\,X}\left(\mathrm{H}_{\mathrm{T}}(Y)\right) = \sum_{|I|\geq 0}\Bigg(&\mathrm{pr}\,X(\mathrm{H}_{\mathrm{T}}^{ab,cd,I})D_I Y_{cd} + \mathrm{H}_{\mathrm{T}}^{ab,cd,I}D_I([X,Y]_{cd}) \\ &+ \sum_{|J|\geq 0}\mathrm{H}_{\mathrm{T}}^{ab,cd,I}(D_J Y_{ef})\partial^{ef,J}Q_{cd,I} + \mathrm{H}_{\mathrm{T}}^{ef,cd,I}(D_I Y_{cd})\partial^{ab}Q_{ef} \\ &+ \frac{\partial P^j}{\partial x^j}\,\mathrm{H}_{\mathrm{T}}^{ab,cd,I}D_I Y_{cd}\Bigg)dg_{ab}\wedge\nu.\end{aligned} \tag{2.18}$$

A comparison of (2.15) with (2.18) yields equation (2.14), as required. QED

Proposition 3. *Let* T *be a third order source form. Then the components* $\mathrm{H}_{\mathrm{T}}^{ab,cd}$, $\mathrm{H}_{\mathrm{T}}^{ab,cd,i}$, $\mathrm{H}_{\mathrm{T}}^{ab,cd,ij}$, $\mathrm{H}_{\mathrm{T}}^{ab,cd,ijk}$ *of the Helmholtz operator associated with* T *satisfy the integrability conditions*

$$\mathrm{H}_{\mathrm{T}}^{ab,cd} + \mathrm{H}_{\mathrm{T}}^{cd,ab} = D_i\mathrm{H}_{\mathrm{T}}^{ab,cd,i} - D_iD_j\mathrm{H}_{\mathrm{T}}^{ab,cd,ij} + D_iD_jD_k\mathrm{H}_{\mathrm{T}}^{ab,cd,ijk}, \tag{2.19a}$$

$$\mathrm{H}_{\mathrm{T}}^{ab,cd,i} - \mathrm{H}_{\mathrm{T}}^{cd,ab,i} = 2D_j\mathrm{H}_{\mathrm{T}}^{ab,cd,ij} - 3D_jD_k\mathrm{H}_{\mathrm{T}}^{ab,cd,ijk}, \tag{2.19b}$$

$$\mathrm{H}_{\mathrm{T}}^{ab,cd,ij} + \mathrm{H}_{\mathrm{T}}^{cd,ab,ij} = 3D_k\mathrm{H}_{\mathrm{T}}^{ab,cd,ijk}, \tag{2.19c}$$

$$\mathrm{H}_{\mathrm{T}}^{ab,cd,ijk} - \mathrm{H}_{\mathrm{T}}^{cd,ab,ijk} = 0. \tag{2.19d}$$

Proof. Equations (2.19) can be derived by direct computation using the expressions (2.13) for $\mathrm{H}_{\mathrm{T}}^{ab,cd,I}$. Since T of order 3, these yield

$$\mathrm{H}_{\mathrm{T}}^{ab,cd,ijk} = \partial^{cd,ijk}\mathrm{T}^{ab} + \partial^{ab,ijk}\mathrm{T}^{cd}, \tag{2.20}$$

which immediately implies equation (2.19d). Likewise, by (2.13),

$$\mathrm{H}_{\mathrm{T}}^{ab,cd,ij} + \mathrm{H}_{\mathrm{T}}^{cd,ab,ij} = 3D_k(\partial^{ab,ijk}\mathrm{T}^{cd} + \partial^{cd,ijk}\mathrm{T}^{ab}) = 3D_k\mathrm{H}_{\mathrm{T}}^{ab,cd,ijk},$$

so that (2.19c) holds true. Next we compute

$$\begin{aligned}
&\mathrm{H}_{\mathrm{T}}^{ab,cd,i} - \mathrm{H}_{\mathrm{T}}^{cd,ab,i}\\
&\quad = 2D_j(\partial^{cd,ij}\mathrm{T}^{ab} - \partial^{ab,ij}\mathrm{T}^{cd}) + 3D_jD_k(\partial^{ab,ijk}\mathrm{T}^{cd} - \partial^{cd,ijk}\mathrm{T}^{ab})\\
&\quad = 2D_j(\partial^{cd,ij}\mathrm{T}^{ab} - \partial^{ab,ij}\mathrm{T}^{cd} + 3D_k\partial^{ab,ijk}\mathrm{T}^{cd}) - 3D_jD_k(\partial^{ab,ijk}\mathrm{T}^{cd} + \partial^{cd,ijk}\mathrm{T}^{ab})\\
&\quad = 2D_j\mathrm{H}_{\mathrm{T}}^{ab,cd,ij} - 3D_jD_k\mathrm{H}_{\mathrm{T}}^{ab,cd,ijk},
\end{aligned}$$

which yields (2.19b). Finally,

$$\begin{aligned}
\mathrm{H}_{\mathrm{T}}^{ab,cd} + \mathrm{H}_{\mathrm{T}}^{cd,ab} &= D_i(\partial^{cd,i}\mathrm{T}^{ab} + \partial^{ab,i}\mathrm{T}^{cd})\\
&\quad - D_iD_j(\partial^{cd,ij}\mathrm{T}^{ab} + \partial^{ab,ij}\mathrm{T}^{cd}) + D_iD_jD_k(\partial^{cd,ijk}\mathrm{T}^{ab} + \partial^{ab,ijk}\mathrm{T}^{cd})\\
&= D_i(\partial^{cd,i}\mathrm{T}^{ab} + \partial^{ab,i}\mathrm{T}^{cd} - 2D_j\partial^{ab,ij}\mathrm{T}^{cd} + 3D_jD_k\partial^{ab,ijk}\mathrm{T}^{cd})\\
&\quad - D_iD_j(\partial^{cd,ij}\mathrm{T}^{ab} - \partial^{ab,ij}\mathrm{T}^{cd} + 3D_k\partial^{ab,ijk}\mathrm{T}^{cd}) + D_iD_jD_k(\partial^{cd,ijk}\mathrm{T}^{ab} + \partial^{ab,ijk}\mathrm{T}^{cd})\\
&= D_i\mathrm{H}_{\mathrm{T}}^{ab,cd,i} - D_iD_j\mathrm{H}_{\mathrm{T}}^{ab,cd,ij} + D_iD_jD_k\mathrm{H}_{\mathrm{T}}^{ab,cd,ijk},
\end{aligned}$$

which proves the first identity (2.19a). QED

Remark 4. The Helmholtz operator H_{T} can also be characterized in terms of the differential δ_V of the variational bicomplex [2] as $\mathrm{H}_{\mathrm{T}} = \delta_V\mathrm{T}$, whereby the foregoing conditions ensue from the general identity $\delta_V^2 = 0$.

The following Lie derivative formula, as established in [2, 7], is central in the proof of our main Theorem.

Proposition 5. *Let* T *be a source form and* X *a projectable vector field on* G*. Then*

$$\mathcal{L}_{\mathrm{pr}\,X}\mathrm{T} = \mathrm{E}(X_{\mathrm{ev}} \lrcorner \mathrm{T}) + \mathrm{H}_{\mathrm{T}}(X_{\mathrm{ev}}). \tag{2.21}$$

An extension of the Lie derivative formula (2.21) to non-projectable, generalized vector fields can be found in [2]. If T is a locally variational source form, then equation (2.21) reduces to

$$\mathcal{L}_{\mathrm{pr}\,X}\mathrm{T} = \mathrm{E}(X_{\mathrm{ev}} \lrcorner \mathrm{T}).$$

Now

$$\mathcal{L}_{\mathrm{pr}\,X}\mathrm{T} = 0,$$

if the source form is invariant under X, that is, if X is a *distinguished symmetry* of the system $\mathrm{T}^{ab} = 0$, while

$$\mathrm{E}(X_{\mathrm{ev}} \lrcorner \mathrm{T}) = 0,$$

provided that X generates a conservation law for T; see [2]. Thus equation (2.21) furnishes a version of the classical Noether's theorem for projectable vector fields expressed directly in terms of the system of differential equations without explicit reference to a Lagrangian.

On the other hand, in the situation of Takens' problem, each vector field X belonging to the Lie algebra Γ is, by prescription, a symmetry of the source form T and generates a conservation law for T. These requirements lead to the conditions

$$\mathrm{H}_{\mathrm{T}}(X_{\mathrm{ev}}) = 0, \qquad \text{for all } X \in \Gamma,$$

on the Helmholtz operator of T. The primary objective in the analysis of Takens' problem is to identify, on mathematical or physical grounds, interesting classes $\mathcal{T}$ of source forms (i.e., differential equations) and symmetry algebras Γ of vector fields so that one will be able to classify all Γ-invariant Helmholtz operators H_{T} with $\mathrm{T} \in \mathcal{T}$ satisfying the conditions $\mathrm{H}_{\mathrm{T}}(X_{\mathrm{ev}}) = 0$, $X \in \Gamma$.

We conclude this section by briefly recalling the construction of the generalized Cotton tensors C_P. For more details and proofs we refer to [5].

Let $\omega^i_j = \Gamma^i_{jk} dx^k$, $i, j = 1, \ldots, m$, denote the connection forms of the Riemannian connection associated with a metric g, and, as usual, write

$$\Omega^i_j = d\omega^i_j + \omega^i_k \wedge \omega^k_j$$

for the associated curvature form. In terms of the components of the Riemannian curvature tensor,

$$\Omega^i_j = \frac{1}{2} R^i{}_{jkl} dx^k \wedge dx^l.$$

Next let

$$tr_r \colon \mathfrak{gl}(m) \to \mathbb{R}, \quad tr_r(A) = \mathrm{trace}(A^r), \quad r = 1, 2, \ldots,$$

denote the elementary polynomials on the Lie algebra $\mathfrak{gl}(m)$ invariant under the adjoint action of the general linear group $Gl(m)$. As is well known [26], any $O(p,q)$, $m = p + q$, invariant polynomial $\mathsf{P} \in I^{2n}(\mathfrak{so}(p,q))$ of homogeneous degree $2n$ on the Lie algebra $\mathfrak{so}(p,q)$ of the orthogonal group $O(p,q)$ can be expressed as $\mathsf{P} = \mathsf{p}(tr_2, tr_4, \ldots, tr_{2s})$ for a uniquely determined polynomial $\mathsf{p} \colon \mathbb{R}^s \to \mathbb{R}$.

Next assume that $m = 4n - 1$. We associate to a polynomial $\mathsf{P} = \mathsf{p}(tr_2, tr_4, \ldots, tr_{2s}) \in I^{2n}(\mathfrak{so}(p,q))$ the second order differential functions $E^{abc}_{\mathsf{P}}(\mathsf{g})$ by

$$E^{abc}_{\mathsf{P}}(\mathsf{g})\nu = g^{ad} \mathsf{P}^c_d(\Omega) \wedge dx^b,$$

where the P^c_d are the components of the derivative of P, that is,

$$\frac{d}{dt} \mathsf{P}(A + tB)|_{t=0} = \mathsf{P}^c_d(A) B^d_c.$$

Now the generalized Cotton tensors are defined by

$$C_{\mathsf{P}}(\mathrm{g}) = C_{\mathsf{P}}^{ab} dg_{ab} \wedge \nu,$$

where the components C_{P}^{ab} are, by definition, given by the covariant derivatives

$$C_{\mathsf{P}}^{ab} = \frac{1}{2} \nabla_c \big(E_{\mathsf{P}}^{acb}(\mathrm{g}) + E_{\mathsf{P}}^{bca}(\mathrm{g}) \big).$$

One can show that $C_{\mathsf{P}}(\mathrm{g})$ is a third order, locally variational natural source form [5]. When $m = 3$ and $\mathsf{P} = tr_2$, the Cotton tensor C_{P} agrees with the well-known Cotton-York tensor, [12, 30].

The following theorem is proved in [5].

Theorem 6. *Let* T *be a natural, locally variational source form. Then* T *can be expressed as*

$$\mathrm{T} = \begin{cases} \mathrm{E}(\lambda), & \text{if } m \equiv 0, 1, 2 \mod 4; \\ \mathrm{E}(\lambda) + C_P(g), & \text{if } m \equiv 3 \mod 4, \end{cases} \tag{2.22}$$

where $\lambda = L\nu$ *is a natural Lagrangian form and* C_P *is the generalized Cotton tensor associated with a uniquely determined* $P \in I^{\frac{m+1}{2}}(\mathfrak{so}(p,q))$.

3 Symmetries and conservation laws

In this section we analyze the relationship between the diffeomorphisms symmetries (1.4) and the differential constraints (1.5) expressing divergence-freeness in metric field theories. As is well known, the commutation formula (2.10), together with Noether's theorems, implies that the Euler-Lagrange expression $\mathrm{T} = \mathrm{E}(\lambda)$ of a Lagrangian form λ with symmetries (1.3) possesses symmetries (1.4) and is, in addition, constrained by the identities (1.5). The following result, which bypasses the Lagrangian and is stated directly in terms of the system of differential equations, is a slight but non-vacuous extension of the above conclusions furnished by the classical Noether's theorems; see [4].

Proposition 7. *Suppose that the source form* $\mathrm{T} = \mathrm{T}^{ab}(g^{[k]}) dg_{ab} \wedge \nu$ *is locally variational. Then* T *is natural,* $\mathcal{L}_{\mathrm{pr}\, X_\xi} \mathrm{T} = 0$ *for all* $X_\xi \in \mathfrak{g}$*, if and only if it is divergence-free,* $D_b \mathrm{T}^{ab} + \Gamma^a_{bc} \mathrm{T}^{bc} = 0$.

Proof. By assumption the Helmholtz operator H_T of T vanishes, and so equation (2.21) reduces to

$$\mathcal{L}_{\mathrm{pr}\, X_\xi} \mathrm{T} = \mathrm{E}(X_{\xi,\mathrm{ev}} \lrcorner\, \mathrm{T}) = -\mathrm{E}^{hk} (2\xi^c_{,a} g_{bc} \mathrm{T}^{ab} + \xi^c g_{ab,c} \mathrm{T}^{ab}) dg_{hk} \wedge \nu. \tag{3.1}$$

Recall that the Euler-Lagrange operator annihilates total derivative expressions, $\mathrm{E}(D_a F) = 0$ for all differential functions $F = F(x^i, g^{[k]})$ and $a = 1, \ldots, m$. This permits us to integrate the right-hand side of (3.1) by parts to conclude that

$$\begin{aligned}
&\mathrm{E}^{hk}(2\xi^c_{,a} g_{bc} \mathrm{T}^{ab} + \xi^c g_{ab,c} \mathrm{T}^{ab}) dg_{hk} \wedge \nu \\
&\qquad = -\mathrm{E}^{hk}(\xi^c (2 g_{bc} D_a \mathrm{T}^{ab} + 2 g_{bc,a} \mathrm{T}^{ab} - \xi^c g_{ab,c} \mathrm{T}^{ab})) dg_{hk} \wedge \nu \\
&\qquad = -2\mathrm{E}^{hk}(\xi^c g_{bc} (D_a \mathrm{T}^{ab} + \Gamma^b_{cd} \mathrm{T}^{cd})) dg_{hk} \wedge \nu.
\end{aligned}$$

Hence

$$\mathcal{L}_{\mathrm{pr}\, X_\xi} \mathrm{T} = 2\mathrm{E}^{hk}(\xi^c g_{bc} (D_a \mathrm{T}^{ab} + \Gamma^b_{cd} \mathrm{T}^{cd})) dg_{hk} \wedge \nu, \tag{3.2}$$

which immediately establishes the Proposition in one direction.

It then remains to prove that the condition $\mathcal{L}_{\mathrm{pr}\, X_\xi} \mathrm{T} = 0$ for all $\xi \in \mathcal{X}(\mathbb{R}^m)$ implies that $D_a \mathrm{T}^{ab} + \Gamma^b_{cd} \mathrm{T}^{cd} = 0$. For this, suppose that for some index b_o, $D_a \mathrm{T}^{ab_o} + \Gamma^{b_o}_{cd} \mathrm{T}^{cd} = F(x^i, g^{[l]})$ is of order l and that for some h, k and J with $|J| = l$,

$$(\partial^{hk,J} F)(x^i_o, g^{[l]}_o) \neq 0, \qquad (x^i_o, g^{[l]}_o) \in J^l(\mathsf{G}).$$

Now choose $\xi^c_o \in \mathcal{X}(\mathbb{R}^m)$ such that

$$\frac{\partial^{|J|} \xi^c_o}{\partial x^J}(x^i_o) g_{o,bc} = \delta_{bb_o}, \qquad \frac{\partial^{|K|} \xi^c_o}{\partial x^K}(x^i_o) g_{o,bc} = 0, \quad K \neq J.$$

Let $(x^i_o, g^{[2l]}_o) \in J^{2l}(\mathsf{G})$ be any point projecting to $g^{[l]}_o$. Then

$$\begin{aligned}
&\mathrm{E}^{hk}(\xi^c_o g_{bc} (D_a \mathrm{T}^{ab} + \Gamma^b_{cd} \mathrm{T}^{cd}))(x^i_o, g^{[2l]}_o) \\
&\quad = \sum_{|I| \leq l} (-1)^{|I|} D_I \left(\partial^{hk,I} (\xi^c_o g_{bc} (D_a \mathrm{T}^{ab} + \Gamma^b_{cd} \mathrm{T}^{cd})) \right) (x^i_o, g^{[2l]}_o) = (\partial^{hk,J} F)(x^i_o, g^{[l]}_o) \neq 0,
\end{aligned}$$

which is a contradiction. Thus $D_a \mathrm{T}^{ab} + \Gamma^b_{cd} \mathrm{T}^{cd} = h^b(x^i)$ are functions of x^i only. But due to translational invariance of the source form T, each h^b must be constant. Finally, by the definition of the total derivative operators (2.3) and the Chrisoffel symbols Γ^b_{cd}, the expressions $D_a \mathrm{T}^{ab} + \Gamma^b_{cd} \mathrm{T}^{cd}$ vanish when evaluated on the jet of the constant metric $\mathsf{g} = \mathrm{diag}(1, \ldots, 1, -1, \ldots, -1)$, showing that $h^b = 0$. QED

Recall that a source form $\mathrm{T} = \mathrm{T}^{ab} dg_{ab} \wedge \nu$ admits translational conservation laws if

$$\mathrm{E}(g_{ab,p} \mathrm{T}^{ab}) = 0, \qquad \text{for all } 1 \leq p \leq m.$$

Proposition 8. *Suppose that a source form* T *is natural and admits translational conservation laws. Then the covariant divergence of* T *vanishes,* $D_b \mathrm{T}^{ab} + \Gamma^a_{bc} \mathrm{T}^{bc} = 0$.

Proof. We will prove this Proposition by showing that

$$\mathrm{H}_\mathrm{T}(X_{\xi,\mathrm{ev}}) = 0 \qquad \text{for all } \xi \in \mathcal{X}(\mathbb{R}^m). \tag{3.3}$$

First, with $X = X_\xi = \xi^a \partial/\partial x^a - 2\xi^c_{,a} g_{bc} \partial^{ab} \in \mathfrak{g}$, equation (2.14) becomes

$$\mathrm{pr}\, X_\xi \mathrm{H}_\mathrm{T}^{ab,cd,I} + \sum_{|J|\geq|I|} (\partial^{cd,I} X_{\xi,ef,J}) \mathrm{H}_\mathrm{T}^{ab,ef,J} - 2\xi^{(a}_{,e} \mathrm{H}_\mathrm{T}^{b)e,cd,I} + \xi^e_{,e} \mathrm{H}_\mathrm{T}^{ab,cd,I} = 0, \tag{3.4}$$

where $X_{\xi,ef,J} = \mathrm{pr}\, X_\xi(g_{ef,J})$. The source form T is also invariant under $\tau_p = \partial/\partial x^p$, so the hypothesis and equation (2.21) imply

$$\sum_{|I|\geq 0} g_{cd,Ip} \mathrm{H}_\mathrm{T}^{ab,cd,I} = 0, \qquad \text{for all } 1 \leq p \leq m. \tag{3.5}$$

Now apply the vector field $\mathrm{pr}\, X_\xi$ to the above equation to see that

$$\sum_{|I|\geq 0} X_{\xi,cd,Ip} \mathrm{H}_\mathrm{T}^{ab,cd,I} + \sum_{|I|\geq 0} g_{cd,Ip}\, \mathrm{pr}\, X_\xi(\mathrm{H}_\mathrm{T}^{ab,cd,I}) = 0. \tag{3.6}$$

Combining (3.4) and (3.6) we obtain

$$\begin{aligned}\sum_{|I|\geq 0} X_{\xi,cd,Ip} \mathrm{H}_\mathrm{T}^{ab,cd,I} - \sum_{|I|\geq 0} g_{cd,Ip} \left(\sum_{|J|\geq|I|} (\partial^{cd,I} X_{\xi,ef,J}) \mathrm{H}_\mathrm{T}^{ab,ef,J} - 2\xi^{(a}_{,e} \mathrm{H}_\mathrm{T}^{b)e,cd,I} + \xi^e_{,e} \mathrm{H}_\mathrm{T}^{ab,cd,I} \right) \\ = \sum_{|I|\geq 0} X_{\xi,cd,Ip} \mathrm{H}_\mathrm{T}^{ab,cd,I} - \sum_{|I|\geq 0} \sum_{|J|\geq|I|} g_{cd,Ip} (\partial^{cd,I} X_{\xi,ef,J}) \mathrm{H}_\mathrm{T}^{ab,ef,J} = 0,\end{aligned}$$

where we used (3.5) twice. On account of the definition of the total derivative operators (2.3), the above equation can be written as

$$\sum_{|I|\geq 0} X_{\xi,cd,Ip} \mathrm{H}_\mathrm{T}^{ab,cd,I} - \sum_{|I|\geq 0} D_p X_{\xi,cd,I} \mathrm{H}_\mathrm{T}^{ab,cd,I} + \sum_{|I|\geq 0} \frac{\partial X_{\xi,cd,I}}{\partial x^p} \mathrm{H}_\mathrm{T}^{ab,cd,I} = 0. \tag{3.7}$$

By the standard prolongation formula (2.6),

$$X_{\xi,cd,Ip} = D_p X_{\xi,cd,I} - \frac{\partial \xi^q}{\partial x^p} g_{cd,Iq}.$$

Thus equation (3.7) simplifies to

$$\begin{aligned}- \sum_{|I|\geq 0} \frac{\partial \xi^q}{\partial x^p} g_{cd,Iq} \mathrm{H}_\mathrm{T}^{ab,cd,I} + \sum_{|I|\geq 0} \frac{\partial X_{\xi,cd,I}}{\partial x^p} \mathrm{H}_\mathrm{T}^{ab,cd,I} \\ = \sum_{|I|\geq 0} \frac{\partial X_{\xi,cd,I}}{\partial x^p} \mathrm{H}_\mathrm{T}^{ab,cd,I} = \sum_{|I|\geq 0} X_{\partial\xi/\partial x^p,cd,I} \mathrm{H}_\mathrm{T}^{ab,cd,I} = 0,\end{aligned}$$

where we again used (3.5). The vector field $\xi \in \mathcal{X}(\mathbb{R}^m)$ is arbitrary, which allows us to conclude that

$$\sum_{|I|\geq 0} X_{\xi,cd,I} \mathrm{H}_{\mathrm{T}}^{ab,cd,I} = 0, \qquad \text{for all } \xi \in \mathcal{X}(\mathbb{R}^m).$$

But on account (3.5),

$$\sum_{|I|\geq 0} X_{\xi,cd,I} \mathrm{H}_{\mathrm{T}}^{ab,cd,I} = \sum_{|I|\geq 0} (D_I X_{\xi,\mathrm{ev},cd} + \xi^q g_{cd,Iq}) \mathrm{H}_{\mathrm{T}}^{ab,cd,I} = \sum_{|I|\geq 0} D_I X_{\xi,\mathrm{ev},cd} \mathrm{H}_{\mathrm{T}}^{ab,cd,I} = 0,$$

that is, equation (3.3) holds.

Due to the $\mathfrak{g}$-invariance of the source form T and condition (3.3), the Lie derivative formula (2.21) now yields

$$\mathrm{E}(\xi^c g_{bc}(D_a \mathrm{T}^{ab} + \Gamma^b_{cd} \mathrm{T}^{cd})) = 0, \qquad \text{for all } \xi^c \in \mathcal{X}(\mathbb{R}^m).$$

Next we continue as in the second part of the proof of Proposition 7 to conclude that the source form T is divergence-free. QED

4 Proof of Theorem 1

The proof of Theorem 1 relies on the following Lemma, which is a special case of a more general result presented in [6, 22].

Lemma 9. *Let* $\mathrm{T} = \mathrm{T}^{ab} dg_{ab} \wedge \nu$, $\mathrm{T}^{ab} = \mathrm{T}^{(ab)}(x^i, g^{[k]})$, *be a* k*-th order source form, where* $k \geq 1$, *and assume that the covariant divergence* $D_b \mathrm{T}^{ab} + \Gamma^a_{bc} \mathrm{T}^{bc} = 0$ *vanishes identically. Then the component functions* T^{ab} *are polynomials in the* k*-th order derivative variables* $g_{cd,i_1\cdots i_k}$ *of degree at most* $m-1$, *where* m *is the number of the independent variables.*

Proof. By assumption,

$$\frac{\partial \mathrm{T}^{ab}}{\partial x^b} + \sum_{|I|\geq 0} g_{cd,Ib} \partial^{cd,I} \mathrm{T}^{ab} + \Gamma^a_{cd} \mathrm{T}^{cd} = 0. \tag{4.1}$$

Now terms in (4.1) involving the order $k+1$ variables $g_{ab,J}$, $|J| = k+1$, yield the equations

$$\partial^{cd,(I} \mathrm{T}^{b)a} = 0. \tag{4.2}$$

Write $\partial^{b\beta}_{k,X}$ for the partial differential operator

$$\partial^{ab}_{k,X} = \sum_{|I|=k} X_I \partial^{ab,I} = \sum \partial^{ab,i_1\cdots i_k} X_{i_1} \cdots X_{i_k}, \tag{4.3}$$

where $X = (X_1, \ldots, X_m) \in (\mathbb{R}^m)^*$ is a covector on $\mathbb{R}^m$. Then equation (4.2) is equivalent to

$$X_b \partial^{cd}_{k,X} \mathrm{T}^{ab} = 0 \qquad \text{for all } X. \tag{4.4}$$

Next let $G^{a,c_1d_1\cdots c_md_m}$ denote the mappings

$$G^{a,c_1d_1\cdots c_md_m}(X^1,\ldots,X^m,Y) = \partial^{c_1d_1}_{k,X^1}\cdots\partial^{c_md_m}_{k,X^m}\mathrm{T}^{ab}Y_b.$$

The operator T^{ab} is polynomial in $g_{ab,I}$, $|I| = k$, of degree at most $m-1$ if and only if all the mappings $G^{a,c_1d_1\cdots c_md_m}$ vanish identically. But by (4.4) and linearity in Y, the equation

$$G^{a,c_1d_1\cdots c_md_m}(X^1,\ldots,X^m,Y) = 0$$

holds whenever Y is a linear combination of the covectors X^1, ..., X^m. Consequently $G^{a,c_1d_1\cdots c_md_m}$ vanishes for almost all X^1, ..., X^m, $Y \in (\mathbb{R}^n)^*$. By continuity, $G^{a,c_1d_1\cdots c_md_m}$ must vanish identically. QED

We next employ the Lie derivative formula (2.21) to derive key identities among the components of the Helmholtz operator associated with a third order natural, divergence free source form. On account of these identities and the integrability conditions of Proposition 3, the proof of the first part of Theorem 1 reduces to showing only that the third order components of the Helmholtz operator vanish.

Proposition 10. *Suppose that* $\mathrm{T} = \mathrm{T}^{ab}dg_{ab}\wedge\nu$ *is a third order natural, divergence free source form, that is,*

$$\mathcal{L}_{\mathrm{pr}\,X_\xi}\mathrm{T} = 0, \quad \textit{for all } X_\xi \in \mathfrak{g}, \quad \textit{and} \quad D_b\mathrm{T}^{ab} + \Gamma^a_{bc}\mathrm{T}^{bc} = 0.$$

Then the components $\mathrm{H}^{ab,cd}_{\mathrm{T}}$, $\mathrm{H}^{ab,cd,i}_{\mathrm{T}}$, $\mathrm{H}^{ab,cd,ij}_{\mathrm{T}}$, $\mathrm{H}^{ab,cd,ijk}_{\mathrm{T}}$ *of the Helmholtz operator* H_{T} *associated with* T *satisfy the following identities.*

$$g_{cd,e}\mathrm{H}^{ab,cd}_{\mathrm{T}} + g_{cd,ie}\mathrm{H}^{ab,cd,i}_{\mathrm{T}} + g_{cd,ije}\mathrm{H}^{ab,cd,ij}_{\mathrm{T}} + g_{cd,ijke}\mathrm{H}^{ab,cd,ijk}_{\mathrm{T}} = 0, \tag{4.5a}$$

$$\begin{aligned}2g_{ce}\mathrm{H}^{ab,cf}_{\mathrm{T}} + 2g_{ce,i}\mathrm{H}^{ab,cf,i}_{\mathrm{T}} + g_{cd,e}\mathrm{H}^{ab,cd,f}_{\mathrm{T}} + 2g_{ce,ij}\mathrm{H}^{ab,cf,ij}_{\mathrm{T}}&\\ +2g_{cd,ie}\mathrm{H}^{ab,cd,if}_{\mathrm{T}} + 2g_{ce,ijk}\mathrm{H}^{ab,cf,ijk}_{\mathrm{T}} + 3g_{cd,ije}\mathrm{H}^{ab,cd,ijf}_{\mathrm{T}} &= 0,\end{aligned} \tag{4.5b}$$

$$\begin{aligned}2g_{ce}\mathrm{H}^{ab,c(d,i)}_{\mathrm{T}} + 4g_{ce,j}\mathrm{H}^{ab,c(d,i)j}_{\mathrm{T}} + g_{cf,e}\mathrm{H}^{ab,cf,di}_{\mathrm{T}}&\\ +6g_{ce,jk}\mathrm{H}^{ab,c(d,i)jk}_{\mathrm{T}} + 3g_{cf,je}\mathrm{H}^{ab,cf,dij}_{\mathrm{T}} &= 0,\end{aligned} \tag{4.5c}$$

$$2g_{ce}\mathrm{H}^{ab,c(d,ij)}_{\mathrm{T}} + 6g_{ce,k}\mathrm{H}^{ab,c(d,ij)k}_{\mathrm{T}} + g_{ck,e}\mathrm{H}^{ab,ck,ijd}_{\mathrm{T}} = 0, \tag{4.5d}$$

$$\mathrm{H}^{ab,c(d,ijk)}_{\mathrm{T}} = 0. \tag{4.5e}$$

Proof. By the Lie derivative formula (2.21),

$$\mathrm{H}_{\mathrm{T}}(X_{\xi,\mathrm{ev}}) = (D_IX_{\xi,\mathrm{ev},cd})\mathrm{H}^{ab,cd,I}_{\mathrm{T}}dg_{ab}\wedge\nu = 0, \tag{4.6}$$

for all $X_\xi \in \mathfrak{g}$, where

$$X_{\xi,\mathrm{ev}} = X_{\xi,\mathrm{ev},cd}\partial^{cd} = -(2\xi^e_{,(c}g_{d)e} + \xi^e g_{cd,e})\partial^{cd}.$$

Let $\mu^{cd,I} = \mu^{(cd),(I)}$, $|I| \geq 0$, be multivectors. We compute

$$D_I(\xi^e_{,c} g_{de})\mu^{cd,I} = \sum_{|I|,|J|\geq 0} \binom{|I|+|J|}{|I|} \xi^e_{,cI} g_{de,J} \mu^{cd,IJ} = \sum_{|I|>0} \xi^e_{,I} \sum_{|J|\geq 0} \binom{|I|+|J|-1}{|I|-1} g_{de,J} \mu^{d(i_1,i_2\cdots i_p)J}, \tag{4.7}$$

and

$$D_I(\xi^e g_{cd,e})\mu^{cd,I} = \sum_{|I|\geq 0} \xi^e_{,I} \sum_{|J|\geq 0} \binom{|I|+|J|}{|I|} g_{cd,Je} \mu^{cd,IJ}. \tag{4.8}$$

The derivatives $\xi^e_{,I}$ are independent, so in light of (4.7), (4.8), the coefficient of ξ^e in (4.6) yields the equation

$$\sum_{|J|\geq 0} g_{cd,Je} \mathrm{H}_{\mathrm{T}}^{ab,cd,J} = 0, \tag{4.9}$$

while for $|I| > 0$, we obtain

$$\sum_{|J|\geq 0} \left[\binom{|I|+|J|-1}{|I|-1} 2 g_{de,J} \mathrm{H}_{\mathrm{T}}^{ab,d(i_1,i_2\cdots i_p)J} + \binom{|I|+|J|}{|I|} g_{cd,Je} \mathrm{H}_{\mathrm{T}}^{ab,cd,IJ} \right] = 0. \tag{4.10}$$

Keeping in mind that $\mathrm{H}_{\mathrm{T}}^{ab,cd,I} = 0$ for $|I| \geq 4$, equations (4.5) follow from (4.9) and (4.10) by inspection. *QED*

We note that if $\mu^{abdc} = \mu^{(ab)(cd)}$ is a valence 4 tensor satisfying the cyclic identity $\mu^{a(bcd)} = 0$, then

$$\mu^{abcd} - \mu^{cdab} = \frac{3}{2}(\mu^{a(bcd)} + \mu^{b(acd)} - \mu^{c(abd)} - \mu^{d(abc)}) = 0, \tag{4.11}$$

so that μ^{abcd} is symmetric under the interchange of the pairs of indices ab and cd.

Corollary 11. *Suppose that* $\mathrm{T} = \mathrm{T}^{ab} dg_{ab} \wedge \nu$ *is a third order natural, divergence free source form. Then* T *is locally variational provided that the third order components* $\mathrm{H}_{\mathrm{T}}^{ab,cd,ijk}$ *of the Helmholtz operator* H_{T} *associated with* T *vanish.*

Proof. We shall show that as a consequence of the assumptions, all the components of the Helmholtz operator H_{T} vanish identically. The proof is based on identities (2.19) and (4.5) of Propositions 3 and 10, respectively.

We first observe that by (2.19c), the second order components $\mathrm{H}_{\mathrm{T}}^{ab,cd,ij}$ are skew-symmetric under the interchange of the index pairs ab and cd. Due to the assumptions, equation (4.5d) reduces to $\mathrm{H}_{\mathrm{T}}^{ab,c(d,ij)} = 0$. Consequently, identity (4.11) applied to the index pairs cd and ij shows that the $\mathrm{H}_{\mathrm{T}}^{ab,cd,ij}$ are also symmetric under the interchange of these pairs. Thus the second order components $\mathrm{H}_{\mathrm{T}}^{ab,cd,ij}$ must vanish identically.

By the above, equation (4.5c) now implies that $\mathrm{H}_{\mathrm{T}}^{ab,c(d,i)} = 0$. Thus $\mathrm{H}_{\mathrm{T}}^{ab,cd,i}$ is symmetric and skew-symmetric in overlapping pairs of indices and hence must vanish. Finally, equation (4.5b) presently shows that $\mathrm{H}_{\mathrm{T}}^{ab,cd} = 0$ for all a, b, c, d. *QED*

We will use a semi-colon to indicate differentiation with respect to the weighted partial derivative operators (2.2), so that, for example,

$$\partial^{ab,ij}\partial^{cd,klm}F = F^{;ab,ij;cd,klm},$$

for a differential function F on $J^\infty(\mathsf{G})$.

We will use the following result in the course of the proof of Theorem 1.

Proposition 12. *Let* $\mathrm{T} = \mathrm{T}^{ab}dg_{ab}\wedge\nu$ *be a third order divergence-free source form and let* $\mathrm{H}_{\mathrm{T}}^{ab,cd,i_1i_2}$ *denote the second order components of the Helmholtz operator associated with* T. *Then*

$$(i) \qquad \mathrm{T}^{ab;c_1c_2,i_1i_2(i_3|;d_1d_2,|j_1j_2j_3)} = \mathrm{T}^{i_1i_2;c_1c_2,ab(i_3|;d_1d_2,|j_1j_2j_3)}; \tag{4.12}$$

$$(ii) \qquad \mathrm{T}^{(ab|;c_1c_2,|i_1)i_2i_3} = \frac{1}{3}\mathrm{T}^{i_2i_3;c_1c_2,abi_1}; \tag{4.13}$$

$$(iii) \qquad \mathrm{H}_{\mathrm{T}}^{ab,c_1c_2,i_1i_2;d_1d_2,j_1j_2j_3j_4} = 3\mathrm{T}^{i_1i_2;ab,c_1c_2(j_1|;d_1d_2,|j_2j_3j_4)}, \tag{4.14}$$

where indices enclosed within vertical bars are omitted in the symmetrization.

Proof. Since T is divergence free and of order 3, equation (4.1) yields

$$\mathrm{T}^{a(b|;c_1c_2|i_1i_2i_3)} = 0. \tag{4.15}$$

Consequently,

$$\mathrm{T}^{a(b|;c_1c_2|i_1i_2)i_3} = -\frac{1}{3}\mathrm{T}^{ai_3;c_1c_2,bi_1i_2}, \tag{4.16}$$

which, on account of (4.15), implies that

$$\begin{aligned}\mathrm{T}^{a(b|;c_1c_2|i_1i_2)(i_3|;d_1d_2,|j_1j_2j_3)} &= -\frac{1}{3}\mathrm{T}^{a(i_3|;c_1c_2,bi_1i_2;d_1d_2,|j_1j_2j_3)}\\ &= -\frac{1}{3}\partial^{c_1c_2,bi_1i_2}\mathrm{T}^{a(i_3|;d_1d_2,|j_1j_2j_3)} = 0.\end{aligned}$$

Equation (4.11) now yields (4.12).

Next, due to (4.15), we have that

$$\mathrm{T}^{(ab|;c_1c_2,|i_1)i_2i_3} = -\operatorname*{Sym}_{\{abi_1\}}\mathrm{T}^{(i_2|a;c_1c_2,bi_1|i_3)} = \frac{1}{3}\mathrm{T}^{i_2i_3;c_1c_2,abi_1}, \tag{4.17}$$

where in the second step we used equation (4.16).

Finally, in light of the commutation formula (2.4),

$$\begin{aligned}\mathrm{H}_{\mathrm{T}}^{ab,c_1c_2,i_1i_2;d_1d_2,j_1j_2j_3j_4} &= \partial^{d_1d_2,j_1j_2j_3j_4}\big(\partial^{c_1c_2,i_1i_2}\mathrm{T}^{ab} - \partial^{ab,i_1i_2}\mathrm{T}^{c_1c_2} + 3D_{i_3}(\partial^{ab,i_1i_2i_3}\mathrm{T}^{c_1c_2})\big)\\ &= 3\mathrm{T}^{c_1c_2;ab,i_1i_2(j_1|;d_1d_2|j_2j_3j_4)} = 3\mathrm{T}^{i_1i_2;ab,c_1c_2(j_1|;d_1d_2|j_2j_3j_4)},\end{aligned}$$

which completes the proof of the Proposition. *QED*

Proof of Theorem 1. Expression (1.6) is a straightforward consequence of Theorem 6 for locally variational source forms. Thus it suffices, by Corollary 11, to show that the third order components $\mathrm{H}_{\mathrm{T}}^{ab,cd,ijk}$ of the Helmholtz operator H_{T} vanish.

In order to streamline our notation, we let $\mathbf{p} = (p_1, p_2)$ and $\mathbf{q} = (q_1, q_2, q_3)$ denote multi-indices of integers $1 \le p_t \le m$, $1 \le q_t \le m$ of respective lengths 2 and 3, and write $\partial^{\mathbf{p},\mathbf{q}} = \partial^{p_1 p_2, q_1 q_2 q_3}$ in what follows. We also employ the notation

$$F^{;\mathbf{p}_1\mathbf{q}_1;\mathbf{p}_2\mathbf{q}_2;\cdots;\mathbf{p}_s\mathbf{q}_s} = \partial^{\mathbf{p}_1,\mathbf{q}_1}\partial^{\mathbf{p}_2,\mathbf{q}_2}\dots\partial^{\mathbf{p}_s,\mathbf{q}_s}F$$

for the repeated derivatives of F with respect to the weighted partial derivative operators $\partial^{\mathbf{p}_u,\mathbf{q}_u} = \partial^{p_{u1}p_{u2},q_{u1}q_{u2}q_{u3}}$.

By Lemma 9, the m-fold derivatives $\mathrm{T}^{ab;\mathbf{p}_1\mathbf{q}_1;\mathbf{p}_2\mathbf{q}_2;\cdots;\mathbf{p}_m\mathbf{q}_m} = 0$, and consequently, the third order components $\mathrm{H}_{\mathrm{T}}^{ab,cd,ijk} = \partial^{cd,ijk}\mathrm{T}^{ab} + \partial^{ab,ijk}\mathrm{T}^{cd}$ of the Helmholtz operator satisfy

$$\mathrm{H}_{\mathrm{T}}^{ab,cd,ijk;\mathbf{p}_1\mathbf{q}_1;\mathbf{p}_2\mathbf{q}_2;\cdots;\mathbf{p}_{m-1}\mathbf{q}_{m-1}} = 0.$$

We will prove by induction that $\mathrm{H}_{\mathrm{T}}^{ab,cd,ijk;\mathbf{p}_1\mathbf{q}_1;\mathbf{p}_2\mathbf{q}_2;\cdots;\mathbf{p}_r\mathbf{q}_r} = 0$ for all $0 \le r \le m-1$.

In order to carry out the induction step we will assume that

$$\mathrm{H}_{\mathrm{T}}^{ab,cd,ijk;\mathbf{p}_1\mathbf{q}_1;\mathbf{p}_2\mathbf{q}_2;\cdots;\mathbf{p}_{r+1}\mathbf{q}_{r+1}} = 0, \qquad \text{for some } 0 \le r \le m-2. \tag{4.18}$$

Our first goal is to show that

$$\mathrm{H}_{\mathrm{T}}^{ab,cd;l_1 l_2,n_1 n_2 n_3 n_4;\mathbf{p}_1\mathbf{q}_1;\mathbf{p}_2\mathbf{q}_2;\cdots;\mathbf{p}_r\mathbf{q}_r} = 0, \tag{4.19a}$$
$$\mathrm{H}_{\mathrm{T}}^{ab,cd,i;l_1 l_2,n_1 n_2 n_3 n_4;\mathbf{p}_1\mathbf{q}_1;\mathbf{p}_2\mathbf{q}_2;\cdots;\mathbf{p}_r\mathbf{q}_r} = 0, \tag{4.19b}$$
$$\mathrm{H}_{\mathrm{T}}^{ab,cd,ij;l_1 l_2,n_1 n_2 n_3 n_4;\mathbf{p}_1\mathbf{q}_1;\mathbf{p}_2\mathbf{q}_2;\cdots;\mathbf{p}_r\mathbf{q}_r} = 0. \tag{4.19c}$$

For this, we start by applying the operator $\partial^{l_1 l_2, n_1 n_2 n_3 n_4}$ to (2.19c) to see that

$$\mathrm{H}_{\mathrm{T}}^{ab,cd,ij;l_1 l_2,n_1 n_2 n_3 n_4} + \mathrm{H}_{\mathrm{T}}^{cd,ab,ij;l_1 l_2,n_1 n_2 n_3 n_4} = 3\mathrm{H}_{\mathrm{T}}^{ab,cd,ij(n_1|;l_1 l_2,|n_2 n_3 n_4)}.$$

Thus by the induction assumption (4.18),

$$\mathrm{H}_{\mathrm{T}}^{ab,cd,ij;l_1 l_2,n_1 n_2 n_3 n_4;\mathbf{p}_1\mathbf{q}_1;\mathbf{p}_2\mathbf{q}_2;\cdots;\mathbf{p}_r\mathbf{q}_r} + \mathrm{H}_{\mathrm{T}}^{cd,ab,ij;l_1 l_2,n_1 n_2 n_3 n_4;\mathbf{p}_1\mathbf{q}_1;\mathbf{p}_2\mathbf{q}_2;\cdots;\mathbf{p}_r\mathbf{q}_r} = 0,$$

that is, the components $\mathrm{H}_{\mathrm{T}}^{ab,cd,ij;l_1 l_2,n_1 n_2 n_3 n_4}$ are skew-symmetric under the interchange of the index pairs ab and cd. On the other hand, repeated differentiation of (4.5d) yields the equation

$$\mathrm{H}_{\mathrm{T}}^{ab,c(d,ij);l_1 l_2,n_1 n_2 n_3 n_4;\mathbf{p}_1\mathbf{q}_1;\mathbf{p}_2\mathbf{q}_2;\cdots;\mathbf{p}_r\mathbf{q}_r} = 0.$$

Thus by (4.11), the components $\mathrm{H}_{\mathrm{T}}^{ab,cd,ij;l_1 l_2,n_1 n_2 n_3 n_4}$ are symmetric in the pairs cd and ij. It now follows that (4.19c) holds.

By differentiating (4.5c) we see that

$$\mathrm{H}_{\mathrm{T}}^{ab,c(d,i);l_1 l_2,n_1 n_2 n_3 n_4;\mathbf{p}_1\mathbf{q}_1;\mathbf{p}_2\mathbf{q}_2;\cdots;\mathbf{p}_r\mathbf{q}_r} = 0,$$

where we used (4.19c) and the identity $H_T^{ab,cd,ijk;l_1l_2,n_1n_2n_3n_4} = 0$. Consequently, (4.19b) holds. Finally, the first equation (4.19a) now follows from (4.5b), again by differentiation.

Next apply the differential operators $\partial^{l_1l_2,n_1n_2n_3n_4}$, $\partial^{\mathbf{p}_1,\mathbf{q}_1}$,..., $\partial^{\mathbf{p}_r,\mathbf{q}_r}$, to equation (4.5a). Keeping in mind that the components $H_T^{ab,cd,ijk}$ are of order 3 we conclude that

$$
\begin{aligned}
g_{cd,e}H_T^{ab,cd;l_1l_2,n_1n_2n_3n_4;\mathbf{p}_1\mathbf{q}_1;\cdots;\mathbf{p}_r\mathbf{q}_r} &+ g_{cd,ie}H_T^{ab,cd,i;l_1l_2,n_1n_2n_3n_4;\mathbf{p}_1\mathbf{q}_1;\cdots;\mathbf{p}_r\mathbf{q}_r} \\
&+ g_{cd,ije}H_T^{ab,cd,ij;l_1l_2,n_1n_2n_3n_4;\mathbf{p}_1\mathbf{q}_1;\cdots;\mathbf{p}_r\mathbf{q}_r} \\
&+ \sum_{t=1}^{r} \delta_e^{(q_{t1}|}H_T^{ab,p_{t1}p_{t2},|q_{t2}q_{t3});l_1l_2,n_1n_2n_3n_4;\mathbf{p}_1\mathbf{q}_1;\cdots\widehat{\mathbf{p}_t\mathbf{q}_t}\cdots;\mathbf{p}_r\mathbf{q}_r} \\
&+ \delta_e^{(n_1|}H_T^{ab,l_1l_2,|n_2n_3n_4);\mathbf{p}_1\mathbf{q}_1;\cdots;\mathbf{p}_r\mathbf{q}_r} = 0,
\end{aligned} \tag{4.20}
$$

where the hat indicates omission. On account of (4.19), the above equation reduces to

$$
\begin{aligned}
\sum_{t=1}^{r} \delta_e^{(q_{t1}|}H_T^{ab,p_{t1}p_{t2},|q_{t2}q_{t3});l_1l_2,n_1n_2n_3n_4;\mathbf{p}_1\mathbf{q}_1;\cdots\widehat{\mathbf{p}_t\mathbf{q}_t}\cdots;\mathbf{p}_r\mathbf{q}_r} \\
+ \delta_e^{(n_1|}H_T^{ab,l_1l_2,|n_2n_3n_4);\mathbf{p}_1\mathbf{q}_1;\cdots;\mathbf{p}_r\mathbf{q}_r} = 0.
\end{aligned} \tag{4.21}
$$

Our next goal is to show that $H_T^{ab,cd,ef;l_1l_2,n_1n_2n_3n_4;\mathbf{p}_1\mathbf{q}_1;\cdots;\mathbf{p}_{r-1}\mathbf{q}_{r-1}}$. As a result, equation (4.21) becomes

$$
\delta_e^{(n_1|}H_T^{ab,l_1l_2,|n_2n_3n_4);\mathbf{p}_1\mathbf{q}_1;\cdots;\mathbf{p}_r\mathbf{q}_r} = 0, \tag{4.22}
$$

which will immediately imply the induction step.

To proceed, we first contract in the index pair e, q_{11} in (4.21), which yields the equation

$$
\begin{aligned}
\frac{m+2}{3}&H_T^{ab,p_{11}p_{12},q_{12}q_{13};l_1l_2,n_1n_2n_3n_4;\mathbf{p}_2\mathbf{q}_2;\cdots;\mathbf{p}_r\mathbf{q}_r} \\
&+ \sum_{t=2}^{r} H_T^{ab,p_{t1}p_{t2},(q_{t1}q_{t2}|;l_1l_2,n_1n_2n_3n_4;\mathbf{p}_1,|q_{t3})q_{12}q_{13};\mathbf{p}_2\mathbf{q}_2;\cdots\widehat{\mathbf{p}_t\mathbf{q}_t}\cdots;\mathbf{p}_r\mathbf{q}_r} \\
&+ H_T^{ab,l_1l_2,(n_1n_2n_3|;\mathbf{p}_1,|n_4)q_{12}q_{13};\mathbf{p}_2\mathbf{q}_2;\cdots;\mathbf{p}_r\mathbf{q}_r} = 0.
\end{aligned} \tag{4.23}
$$

On account of (4.14), we have

$$
H_T^{ab,p_{11}p_{12},q_{12}q_{13};l_1l_2,n_1n_2n_3n_4;\mathbf{p}_2\mathbf{q}_2;\cdots;\mathbf{p}_r\mathbf{q}_r} = 3T^{q_{12}q_{13};ab,p_{11}p_{12}(n_1|;l_1l_2,|n_2n_3n_4);\mathbf{p}_2\mathbf{q}_2;\cdots;\mathbf{p}_r\mathbf{q}_r}.
$$

We next use (4.13) and (4.14) to compute

$$
\begin{aligned}
H_T^{ab,p_{t1}p_{t2},(q_{t1}q_{t2}|;l_1l_2,n_1n_2n_3n_4;\mathbf{p}_1,|q_{t3})q_{12}q_{13};\mathbf{p}_2\mathbf{q}_2;\cdots\widehat{\mathbf{p}_t\mathbf{q}_t}\cdots;\mathbf{p}_r\mathbf{q}_r} \\
&= 3 \operatorname*{Sym}_{\{n_1\cdots n_4\}} T^{(q_{t1}q_{t2}|;ab,p_{t1}p_{t2}n_1;l_1l_2,n_2n_3n_4;\mathbf{p}_1,|q_{t3})q_{12}q_{13};\mathbf{p}_2\mathbf{q}_2;\cdots\widehat{\mathbf{p}_t\mathbf{q}_t}\cdots;\mathbf{p}_r\mathbf{q}_r} \\
&= 3T^{(q_{t1}q_{t2}|;\mathbf{p}_1,|q_{t3})q_{12}q_{13};ab,p_{t1}p_{t2}(n_1|;l_1l_2,|n_2n_3n_4);\mathbf{p}_2\mathbf{q}_2;\cdots\widehat{\mathbf{p}_t\mathbf{q}_t}\cdots;\mathbf{p}_r\mathbf{q}_r} \\
&= T^{q_{12}q_{13};\mathbf{p}_1,\mathbf{q}_t;ab,p_{t1}p_{t2}(n_1|;l_1l_2,|n_2n_3n_4);\mathbf{p}_2\mathbf{q}_2;\cdots\widehat{\mathbf{p}_t\mathbf{q}_t}\cdots;\mathbf{p}_r\mathbf{q}_r}.
\end{aligned}
$$

Similarly, by (2.20) and (4.12),

$$
\begin{aligned}
&\mathrm{H}_{\mathrm{T}}^{ab,l_1l_2,(n_1n_2n_3|;\mathbf{p}_1,|n_4)q_{12}q_{13};\mathbf{p}_2\mathbf{q}_2;\cdots;\mathbf{p}_r\mathbf{q}_r} \\
&\qquad = \mathrm{T}^{ab;l_1l_2,(n_1n_2n_3|;\mathbf{p}_1,|n_4)q_{12}q_{13};\mathbf{p}_2\mathbf{q}_2;\cdots;\mathbf{p}_r\mathbf{q}_r} + \mathrm{T}^{l_1l_2;ab,(n_1n_2n_3|;\mathbf{p}_1,|n_4)q_{12}q_{13};\mathbf{p}_2\mathbf{q}_2;\cdots;\mathbf{p}_r\mathbf{q}_r} \\
&\qquad = \mathrm{T}^{q_{12}q_{13};l_1l_2,(n_1n_2n_3|;\mathbf{p}_1,|n_4)ab;\mathbf{p}_2\mathbf{q}_2;\cdots;\mathbf{p}_r\mathbf{q}_r} + \mathrm{T}^{q_{12}q_{13};ab,(n_1n_2n_3|;\mathbf{p}_1,|n_4)l_1l_2;\mathbf{p}_2\mathbf{q}_2;\cdots;\mathbf{p}_r\mathbf{q}_r}.
\end{aligned}
$$

By the above, equation (4.23) becomes

$$
\begin{aligned}
(m+2)&\mathrm{T}^{q_{12}q_{13};ab,p_{11}p_{12}(n_1|;l_1l_2,|n_2n_3n_4);\mathbf{p}_2\mathbf{q}_2;\cdots;\mathbf{p}_r\mathbf{q}_r} \\
&+ \sum_{t=2}^{r} \mathrm{T}^{q_{12}q_{13};ab,p_{t1}p_{t2}(n_1|;l_1l_2,|n_2n_3n_4);\mathbf{p}_1\mathbf{q}_t;\mathbf{p}_2\mathbf{q}_2;\cdots\widehat{\mathbf{p}_t\mathbf{q}_t}\cdots;\mathbf{p}_r\mathbf{q}_r} \\
&+ \mathrm{T}^{q_{12}q_{13};l_1l_2,(n_1n_2n_3|;\mathbf{p}_1,|n_4)ab;\mathbf{p}_2\mathbf{q}_2;\cdots;\mathbf{p}_r\mathbf{q}_r} \\
&+ \mathrm{T}^{q_{12}q_{13};ab,(n_1n_2n_3|;\mathbf{p}_1,|n_4)l_1l_2;\mathbf{p}_2\mathbf{q}_2;\cdots;\mathbf{p}_r\mathbf{q}_r} = 0.
\end{aligned}
\tag{4.24}
$$

We have thus reduced (4.23) into an equation for the derivatives of a single component $\mathrm{T}^{q_{12}q_{13}}$ of the source form T.

Now assume that equations (4.24) admitted a non-trivial solution in the derivatives of $\mathrm{T}^{q_{12}q_{13}}$. Choose a term $\mathrm{T}^{q_{12}q_{13};ab,p_{11}p_{12}(n_1|;l_1l_2,|n_2n_3n_4);\mathbf{p}_2\mathbf{q}_2;\cdots;\mathbf{p}_r\mathbf{q}_r}$ with maximal absolute value amongst the symmetrized derivatives. Keeping in mind that $r \leq m-2$ by the induction assumption, equation (4.24) implies that

$$
\begin{aligned}
&\left|\mathrm{T}^{q_{12}q_{13};ab,p_{11}p_{12}(n_1|;l_1l_2,|n_2n_3n_4);\mathbf{p}_2\mathbf{q}_2;\cdots;\mathbf{p}_r\mathbf{q}_r}\right| \\
&\leq \frac{1}{m+2}\Bigg(\sum_{t=2}^{r} \left|\mathrm{T}^{q_{12}q_{13};ab,p_{t1}p_{t2}(n_1|;l_1l_2,|n_2n_3n_4);\mathbf{p}_1\mathbf{q}_t;\mathbf{p}_2\mathbf{q}_2;\cdots\widehat{\mathbf{p}_t\mathbf{q}_t}\cdots;\mathbf{p}_r\mathbf{q}_r}\right| \\
&+ \left|\mathrm{T}^{q_{12}q_{13};l_1l_2,(n_1n_2n_3|;\mathbf{p}_1,|n_4)ab;\mathbf{p}_2\mathbf{q}_2;\cdots;\mathbf{p}_r\mathbf{q}_r}\right| + \left|\mathrm{T}^{q_{12}q_{13};ab,(n_1n_2n_3|;\mathbf{p}_1,|n_4)l_1l_2;\mathbf{p}_2\mathbf{q}_2;\cdots;\mathbf{p}_r\mathbf{q}_r}\right| \Bigg) \\
&\leq \frac{m-1}{m+2}\left|\mathrm{T}^{q_1^2q_1^3;ab,p_1^1p_1^2(n_1|;l_1l_2,|n_2n_3n_4);\mathbf{p}_2\mathbf{q}_2;\cdots;\mathbf{p}_r\mathbf{q}_r}\right|.
\end{aligned}
$$

Thus necessarily

$$
\mathrm{H}_{\mathrm{T}}^{ab,cd,ef;l_1l_2,n_1n_2n_3n_4;\mathbf{p}_1\mathbf{q}_1;\cdots;\mathbf{p}_{r-1}\mathbf{q}_{r-1}} = \mathrm{T}^{ef;ab,cd(n_1|;l_1l_2,|n_2n_3n_4);\mathbf{p}_1\mathbf{q}_1;\cdots;\mathbf{p}_{r-1}\mathbf{q}_{r-1}} = 0, \tag{4.25}
$$

for all $a, b, c, d, e, f, l_1, l_2, n_1, \dots, n_4$, $\mathbf{p}_1, \mathbf{q}_1, \dots, \mathbf{p}_{r-1}, \mathbf{q}_{r-1}$. Hence (4.21) reduces to

$$
\delta_e^{(n_1|}\mathrm{H}_{\mathrm{T}}^{ab,l_1l_2,|n_2n_3n_4);\mathbf{p}_1\mathbf{q}_1;\cdots;\mathbf{p}_r\mathbf{q}_r} = 0,
$$

which immediately implies that $\mathrm{H}_{\mathrm{T}}^{ab,l_1l_2,n_1n_2n_3;\mathbf{p}_1\mathbf{q}_1;\cdots;\mathbf{p}_r\mathbf{q}_r} = 0$, completing the induction step.

In conclusion, the third order components $\mathrm{H}_{\mathrm{T}}^{ab,cd,jkl}$ of the Helmholtz operator associated with T vanish identically. By Corollary 11 the Helmholtz operator H_{T} must also vanish identically, and hence T is locally variational. *QED*

5 Discussion

In this paper we prove that a system of third order natural, divergence-free differential equations for the components of the metric tensor can always be written as the Euler-Lagrange expression of some Lagrangian function. Moreover, by the solution of the equivariant inverse problem of the calculus of variations for metric field theories presented in [5], the Lagrangian can be chosen to be natural when the dimension of the underlying space is $m = 0, 1, 2 \mod 4$, and, in dimensions $m = 3 \mod 4$, the system can be written as a sum of the Euler-Lagrange expression of a natural Lagrangian and a generalized Cotton tensor.

Extending our result to fourth order operators either by showing that natural, divergence free systems are necessarily locally variational or by finding non-variational examples of such systems remains a challenging open problem. In contrast, for vector field theories [8] and, in more generality, for Yang-Mills theories [21], non-variational third order source forms with the prescribed symmetries and conservation laws can be derived by the way of natural constructions utilizing the intrinsic geometry of the problem. Note that unlike in the situation of the present paper, proving that a 4th order source form T fulfills the highest order non-vacuous Helmholtz conditions $\mathrm{H}_{\mathrm{T}}^{ab,cd,i_1i_2i_3i_4} = 0$ will not be sufficient to guarantee that T is locally variational.

A significant generalization of the present work would be to enlarge the symmetry group to include conformal transformations of the metric with the resultant condition that the source form be trace free. The problem closely bears on conformal gravity, and, as exemplified by the Bach equations [10], will require the analysis of fourth order systems. A solution for the equivariant inverse problem of the calculus of variations in this situation would also be of substantial independent interest. Additional on physical grounds intriguing problems would be Takens' question for combined field theories involving metric, such as Einstein-Yang-Mills equations in both the abelian and non-abelian cases.

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