

Gauge invariance, charge conservation, and variational principles

Gianni Manno*

Department of Mathematics “E. De Giorgi”,
Università del Salento, via per Arnesano, 73100 Lecce, Italy

`gianni.manno@unile.it`

Juha Pohjanpelto[†]

Department of Mathematics, Oregon State University,
Corvallis, Oregon 97331–4605, U.S.A.

`juha@math.oregonstate.edu`, <http://oregonstate.edu/~pohjanpp/>

Raffaele Vitolo*

Department of Mathematics “E. De Giorgi”,
Università del Salento, via per Arnesano, 73100 Lecce, Italy
`raffaele.vitolo@unile.it`, <http://poincare.unile.it/vitolo/>

<p><i>Journal of Geometry and Physics</i> 58 (2008) 996–1006</p>
--

Abstract

We present new results on the correspondence between symmetries, conservation laws and variational principles for field equations in general non-abelian gauge theories. Our main result states that second order field equations possessing translational and gauge symmetries and the corresponding conservation laws are always derivable from a variational principle. We also show by the way of examples that the above result fails in general for third order field equations.

Keywords: gauge theories, symmetries, conservation laws, variational principles.

MSC 2000 classification: 58E30, 58J70, 70S10, 70S15.

*Supported by PRIN 2005/2007 “Simmetrie e Supersimmetrie Classiche e Quantistiche”, GNSAGA, Università del Salento.

[†]Supported in part by NSF Grants DMS 04–53304, OCE 06–21134, by PRIN 2005/2007 “Simmetrie e Supersimmetrie Classiche e Quantistiche”, Università del Salento, and by Programma Professori Visitatori of GNSAGA of INdAM.

1 Introduction and main results

The classical Noether's theorem establishes essentially a one-to-one correspondence between the symmetries and conservation laws of a system of partial differential equations admitting a variational principle. In 1977, F. Takens [28] considered and rigorously formulated the following novel aspect of Noether's theorem: Let \mathfrak{g} be a Lie algebra of vector fields defined on the space of independent and dependent variables, and suppose that a system of differential equations is invariant under \mathfrak{g} and that each element in \mathfrak{g} generates a conservation law for the system. Does it then follow that the system arises from a variational principle, i.e., that it is the Euler-Lagrange expression of some Lagrangian function? In his original paper Takens considered the question for second order scalar equations, systems of linear equations, and metric field theories. Subsequently, Takens' results on second order scalar equations and on systems of linear equations were substantially generalized by Anderson and Pohjanpelto [3], [4], [26]. We refer to [3] in particular for more background material and motivation on Takens' problem.

Apart from the papers listed above, the literature dealing with the existence of variational principles for systems of differential equations admitting a Lie algebra of symmetries and the corresponding conservation laws is mainly limited to classical field theories, where the symmetry group is the infinite dimensional group of coordinate transformations of the underlying manifold and the conservation laws express the vanishing of the covariant divergence (or some variant of it) of the field equations. The classification results of Cartan [7], Vermeil [29], and Weyl [30] imply that second order quasi-linear field equations for the metric tensor possessing the symmetries and conservation laws of the Einstein equations necessarily arise from a variational principle. This result was later generalized to general second order equations for the metric tensor and to third order equations in the 3-dimensional case by Lovelock, [19, 21], whose results were subsequently extended to metric-scalar [13], metric-vector [20], and metric-bivector [22] theories. More recently, pure vector field theories with the symmetry group consisting of spatial translations and $U(1)$ gauge transformations were treated in [5].

In this paper we investigate the relationship between symmetries, conservation laws, and variational principles for gauge theories with a general structure group on the n -dimensional Euclidean space \mathbb{R}^n . Gauge theories with a non-Abelian structure group play a central role in quantum field theories by providing a unified framework for the description of electromagnetism and the weak and strong forces [10], and in geometry as the pivotal ingredient in the Donaldson and Seiberg-Witten theories [8, 23]. Recently, higher order gauge theories, i.e., theories with field equations of order $k \geq 3$, have been developed in classical [9], [17] as well as in quantum settings [11], [12, p. 217].

The primary goal of the present paper is to identify conditions under which a system of gauge field equations admitting translational and gauge symmetries and the associated conservation laws arises from a variational principle. In contrast to the results of Cartan, Vermeil, Weyl, and Lovelock, field equations of gauge theories, even in low orders, are not specified by these conditions, which rules out solving Takens' question in this situation by a brute force classification process. However, as is well known, the vanishing of the classical Helmholtz conditions for a system of differential equations

guarantees the existence of a Lagrangian for the system, and our problem is rendered tractable by an analysis of these conditions for systems with the prescribed symmetries and conservation laws.

Write (x^i) for the coordinates on \mathbb{R}^n and let $\{e_\alpha\}$ be a basis of the Lie algebra \mathfrak{g} of an r -dimensional Lie group G with structure constants $c_{\beta\gamma}^\alpha$. A gauge field

$$A = A_a^\alpha(x^i)dx^a \otimes e_\alpha$$

is a \mathfrak{g} -valued 1-form on \mathbb{R}^n , where the A_a^α stand for the components of A . The gauge field A is subject to a system of k -th order partial differential equations

$$T_\alpha^a = T_\alpha^a(x^i, A_b^\beta, A_{b,j_1}^\beta, A_{b,j_1j_2}^\beta, \dots, A_{b,j_1j_2\dots j_k}^\beta) = 0, \quad a = 1, \dots, n, \quad \alpha = 1, \dots, r,$$

where $A_{b,j_1j_2\dots j_l}^\beta$ denotes the derivative of A_b^β with respect to the independent variables $x^{j_1}, x^{j_2}, \dots, x^{j_l}$. The operator T_α^a is locally variational if it can be written as the Euler-Lagrange expression

$$T_\alpha^a = E_\alpha^a(L) = \frac{\partial L}{\partial A_a^\alpha} - D_{i_1} \frac{\partial L}{\partial A_{a,i_1}^\alpha} + D_{i_1} D_{i_2} \frac{\partial L}{\partial A_{a,i_1 i_2}^\alpha} - \dots$$

of some locally defined Lagrangian $L = L(x^j, A_b^\beta, A_{b,j_1}^\beta, A_{b,j_1j_2}^\beta, \dots, A_{b,j_1j_2\dots j_l}^\beta)$, where D_i denotes the standard coordinate total derivative operator.

In this paper we consider the following classes of symmetries and conservation laws for a differential operator T_α^a .

Symmetries

[S1] The operator T_α^a is invariant under the infinitesimal group of spatial translations

$$\mathfrak{t}(n) = \left\{ \mathfrak{t} = a^i \frac{\partial}{\partial x^i} \mid (a^i) \in \mathbb{R}^n \right\}. \quad (1)$$

[S2] The operator T_α^a is invariant under the infinite dimensional group of infinitesimal gauge transformations

$$\mathfrak{ga}(n) = \left\{ Q_\varphi = (\varphi_{,a}^\alpha + c_{\beta\gamma}^\alpha A_a^\beta \varphi^\gamma) \frac{\partial}{\partial A_a^\alpha} \mid \varphi \in C^\infty(\mathbb{R}^n, \mathfrak{g}) \right\}. \quad (2)$$

Conservation laws

[C1] There are functions $t_p^i = t_p^i(x^j, A_b^\beta, A_{b,j_1}^\beta, A_{b,j_1j_2}^\beta, \dots, A_{b,j_1j_2\dots j_l}^\beta)$ such that, for each $p = 1, 2, \dots, n$,

$$A_{a,p}^\alpha T_\alpha^a = D_i(t_p^i).$$

[C2] The covariant divergence of the operator T_α^a vanishes identically,

$$\nabla_a T_\alpha^a = D_a T_\alpha^a + c_{\alpha\beta}^\gamma A_a^\beta T_\gamma^a = 0.$$

Our main result is the following.

Theorem 1. *Suppose that the differential operator T_α^a has symmetries [S1], [S2] and conservation laws [C1]. Then T_α^a is locally variational if*

1. T_α^a is of second order, or
2. T_α^a is polynomial in the components of the gauge field and their derivatives of degree at most n .

This paper is organized as follows. After covering some preliminary material relevant to the problem at hand in section 2, in section 3 we analyze the relationship between symmetries [S1], [S2] and conservation laws [C1], [C2] for gauge field equations. In particular, we show that any differential operator T_α^a possessing symmetries [S1], [S2] and conservation laws [C1] also necessarily admits conservation laws [C2]. This interesting though elementary fact does not seem to have been previously noted in the literature. In section 4 we present the proof of Theorem 1, and, finally, in section 5, for $n \geq 3$, we employ a general construction to derive examples of third-order differential operators with symmetries [S1], [S2] and conservation laws [C1], [C2] that fail to be locally variational, showing that Theorem 1 is sharp as far as the order of the differential operator T_α^a is concerned.

2 Preliminaries

In this section we collect together some basic definitions and results from the formal calculus of variations on jet spaces most relevant to the problem at hand. For more details and proofs we refer, e.g., to [1, 25].

Let G be an r -dimensional Lie group with Lie algebra \mathfrak{g} . We write $\mathcal{A} \rightarrow \mathbb{R}^n$ for the bundle of gauge fields with structure group G over the n -dimensional Euclidean space \mathbb{R}^n . Fix coordinates (x^i) on \mathbb{R}^n and a basis $\{e_\alpha\}$ for \mathfrak{g} , and let A_a^α , $a = 1, \dots, n$, $\alpha = 1, \dots, r$, denote the components of the gauge field. Then, as a coordinate bundle, $\mathcal{A} = \{(x^i, A_a^\alpha)\} \rightarrow \{(x^i)\}$. We denote the bundle of order k , $0 \leq k \leq \infty$, jets of local sections of \mathcal{A} by $J^k(\mathcal{A})$; in coordinates

$$J^k(\mathcal{A}) = \{(x^i, A_a^\alpha, A_{a,j_1}^\alpha, A_{a,j_1 j_2}^\alpha, \dots, A_{a,j_1 j_2 \dots j_l}^\alpha, \dots, A_{a,j_1 j_2 \dots j_k}^\alpha)\}, \quad (3)$$

where $A_{a,j_1 j_2 \dots j_l}^\alpha$ stands for the l th order derivative variables. We also use the notation $A^{[l]}$ to collectively denote all variables $A_{a,j_1 j_2 \dots j_p}^\alpha$, $p = 0, \dots, l$, up to order l .

Let $I = (i_1, i_2, \dots, i_k)$, $1 \leq i_l \leq n$, denote an unordered multi-index of length $|I| = k$. Define partial derivative operators $\partial_\alpha^{a,I}$ by

$$\partial_\alpha^{a,I} A_{b,J}^\beta = \begin{cases} \delta_\alpha^\beta \delta_b^a \delta_{j_1}^{i_1} \delta_{j_2}^{i_2} \dots \delta_{j_k}^{i_k}, & \text{if } |I| = |J|, \\ 0, & \text{if } |I| \neq |J|, \end{cases} \quad (4)$$

where round brackets indicate symmetrization in the enclosed indices. Then the standard total derivative operators D_i are given by

$$D_i = \frac{\partial}{\partial x^i} + \sum_{|I| \geq 0} A_{a,i}^\alpha \partial_\alpha^{a,I}, \quad i = 1, 2, \dots, n. \quad (5)$$

When there is no danger of confusion we will employ the standard Einstein summation convention in the Euclidean space and Lie algebra indices.

The flow of a vector field

$$X = P^i(x^j, A_b^\beta) \frac{\partial}{\partial x^i} + Q_a^\alpha(x^j, A_b^\beta) \partial_\alpha^a \quad (6)$$

on \mathcal{A} induces a transformation on the space of section of \mathcal{A} , and, consequently, by differentiation, it induces a local 1-parameter transformation group acting on $J^k(\mathcal{A})$, $k \geq 0$. The associated infinitesimal generator is called the k^{th} -order prolongation of X and is denoted by $\text{pr}^k X$. The components of $\text{pr}^k X$ are given by the usual prolongation formula

$$\text{pr}^k X = P^i D_i + \sum_{|I| \leq k} D_I(X_{\text{ev}_a}^\alpha) \partial_\alpha^{a,I}, \quad (7)$$

where the $X_{\text{ev}_a}^\alpha$ denote the components of the evolutionary form

$$X_{\text{ev}} = (Q_a^\alpha - P^i A_{a,i}^\alpha) \partial_\alpha^a$$

of X and where $D_I = D_{i_1} \cdots D_{i_k}$ for a multi-index $I = (i_1, \dots, i_k)$. We will also write $\text{pr}^\infty X = \text{pr} X$. The vector field (6) is called projectable if the coefficients $P^i = P^i(x^j)$ are functions of the independent variables x^j only. In particular, the infinitesimal generators of translations (1) and gauge transformations (2) form the Lie algebra $\mathfrak{t}(n) \times_s \mathfrak{ga}(n)$ of projectable vector fields acting on \mathbb{R}^n

Given a differential operator $T_\alpha^a = T_\alpha^a(x^i, A^{[k]})$, we associate to it the source form

$$T = T_\alpha^a dA_a^\alpha \wedge \nu, \quad (8)$$

where $\nu = dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n$ is the volume form on \mathbb{R}^n . A vector field X on \mathcal{A} generates a conservation law for the source form T if there are differential functions $t^i = t^i(x^j, A^{[l]})$, $i = 1, \dots, n$, so that

$$X_{\text{ev}_a}^\alpha T_\alpha^a = D_i t^i. \quad (9)$$

The source form T is said to arise from a variational principle if there is a Lagrangian function $L = L(x^i, A^{[l]})$ such that T is the Euler-Lagrange expression of L , i.e.,

$$T_\alpha^a = E_\alpha^a(L) = \sum_{|I| \geq 0} (-1)^{|I|} D_I(\partial_\alpha^{a,I} L). \quad (10)$$

We will call $\lambda = L(x^i, A^{[l]}) \nu$ a Lagrangian n -form and $E(\lambda) = E_\alpha^a(L) dA_a^\alpha \wedge \nu$ the Euler-Lagrange form associated with λ . As is well known, the Euler-Lagrange operator

commutes with the prolonged action of projectable transformations on \mathcal{A} ; infinitesimally,

$$\mathbb{E}(\mathcal{L}_{\text{pr } X} \lambda) = \mathcal{L}_{\text{pr } X} \mathbb{E}(\lambda) \quad (11)$$

for every projectable vector field X and Lagrangian form λ , where \mathcal{L} denotes the standard Lie derivative operator.

As in [5], we define the Helmholtz operator \mathbb{H}_T acting on evolutionary vector fields $Y = Y_a^\alpha \partial_\alpha^a$ on \mathcal{A} by

$$\mathbb{H}_T(Y) = \mathcal{L}_{\text{pr } Y} T - \mathbb{E}(Y \lrcorner T). \quad (12)$$

The components $\mathbb{H}_{\alpha\beta}^{ab,I}$ of \mathbb{H}_T are determined by

$$\mathbb{H}_T(Y) = \sum_{|I| \geq 0} (D_I Y_b^\beta) \mathbb{H}_{\alpha\beta}^{ab,I} dA_a^\alpha \wedge \nu, \quad (13)$$

and are explicitly given by

$$\mathbb{H}_{\alpha\beta}^{ab,I} = \partial_\beta^{b,I} T_\alpha^a - (-1)^{|I|} \mathbb{E}_\alpha^{a,I}(T_\beta^b), \quad |I| \geq 0, \quad (14)$$

where $\mathbb{E}_\alpha^{a,I}$ denotes the higher Euler-Lagrange operators acting on a function F defined on some $J^k(\mathcal{A})$ by

$$\mathbb{E}_\alpha^{a,I}(F) = \sum_{|J| \geq 0} (-1)^{|J|} \binom{|I|+|J|}{|I|} D_J(\partial_\alpha^{a,IJ} F), \quad |I| \geq 0.$$

Note that if $T = T_\alpha^a(x^i, A^{[k]}) dA_a^\alpha \wedge \nu$ is of order k , then $\mathbb{H}_{\alpha\beta}^{ab,I} = 0$ for $|I| > k$ and that for $|I| = 0, \dots, k$, the components $\mathbb{H}_{\alpha\beta}^{ab,I}$ are of order at most $2k - |I|$.

As is well known, a source form $T = \mathbb{E}(L)$ arising from a variational principle satisfies the Helmholtz conditions $\mathbb{H}_T = 0$, or in components, $\mathbb{H}_{\alpha\beta}^{ab,I} = 0$. Conversely, one can show that if the Helmholtz conditions $\mathbb{H}_T = 0$ are satisfied, then, at least locally, the source form T can be written as the Euler-Lagrange expression of some Lagrangian L ; see [1]. We will thus call a source form satisfying the Helmholtz conditions locally variational.

Proposition 2. *Suppose that $X = P^i \partial / \partial x^i + Q_a^\alpha \partial_\alpha^a$ is a projectable vector field on \mathcal{A} and that the source form T is invariant under the prolongation $\text{pr } X$ of X . Then the components $\mathbb{H}_{\alpha\beta}^{ab,I}$ of the Helmholtz operator \mathbb{H}_T of T satisfy the invariance conditions*

$$\mathcal{L}_{\text{pr } X} \mathbb{H}_{\alpha\beta}^{ab,I} + \sum_{|J| \geq 0} \mathbb{H}_{\alpha\gamma}^{ac,J} (\partial_\beta^{b,I} Q_{c,J}^\gamma) + \mathbb{H}_{\gamma\beta}^{cb,I} \partial_\alpha^a Q_c^\gamma + \frac{\partial P^j}{\partial x^j} \mathbb{H}_{\alpha\beta}^{ab,I} = 0, \quad (15)$$

where $Q_{c,J}^\gamma$ stands for the $A_{c,J}^\gamma$ -component of $\text{pr } X$.

Proof. We first compute

$$\begin{aligned}\mathcal{L}_{\text{pr } X}(\mathbb{H}_T(Y)) &= \mathcal{L}_{\text{pr } X}(\mathcal{L}_{\text{pr } Y}T) - \mathcal{L}_{\text{pr } X}\mathbb{E}(Y \lrcorner T) \\ &= \mathcal{L}_{\text{pr}[X, Y]}T - \mathbb{E}((\mathcal{L}_{\text{pr } X}Y) \lrcorner T) = \mathbb{H}_T([X, Y]),\end{aligned}\tag{16}$$

where we used the invariance of T and the fact that the Euler-Lagrange operator is equivariant under the action of the prolongation of a projectable vector field X . Next write $Y = Y_a^\alpha \partial_\alpha^a$. Then on account of (13),

$$\begin{aligned}\mathcal{L}_{\text{pr } X}(\mathbb{H}_T(Y)) &= \sum_{|I| \geq 0} \left((\mathcal{L}_{\text{pr } X} \mathbb{H}_{\alpha\beta}^{ab, I}) D_I Y_b^\beta + \mathbb{H}_{\alpha\beta}^{ab, I} \mathcal{L}_{\text{pr } X}(D_I Y_b^\beta) \right. \\ &\quad \left. + \mathbb{H}_{\gamma\beta}^{cb, I} (D_I Y_b^\beta) \partial_\alpha^a Q_c^\gamma + \frac{\partial P^j}{\partial x^j} \mathbb{H}_{\alpha\beta}^{ab, I} D_I Y_b^\beta \right) dA_a^\alpha \wedge \nu.\end{aligned}\tag{17}$$

But

$$\begin{aligned}\mathcal{L}_{\text{pr } X} D_I Y_b^\beta &= \mathcal{L}_{\text{pr } X}(\text{pr } Y \lrcorner \theta_{b, I}^\beta) \\ &= \text{pr}[X, Y] \lrcorner \theta_{b, I}^\beta + \text{pr } Y \lrcorner (\mathcal{L}_{\text{pr } X} \theta_{b, I}^\beta) \\ &= D_I[X, Y]_b^\beta + \sum_{|J| \geq 0} (D_J Y_c^\gamma) (\partial_\gamma^{c, J} Q_{b, I}^\beta),\end{aligned}\tag{18}$$

where, as usual, $\theta_{b, I}^\beta = dA_{b, I}^\beta - A_{b, I i}^\beta dx^i$ denotes a basic contact form on $J^\infty(\mathcal{A})$. Now by virtue of (18), equation (17) becomes

$$\begin{aligned}\mathcal{L}_{\text{pr } X}(\mathbb{H}_T(Y)) &= \sum_{|I| \geq 0} \left((\mathcal{L}_{\text{pr } X} \mathbb{H}_{\alpha\beta}^{ab, I}) D_I Y_b^\beta + \mathbb{H}_{\alpha\beta}^{ab, I} D_I[X, Y]_b^\beta \right. \\ &\quad \left. + \sum_{|J| \geq 0} \mathbb{H}_{\alpha\gamma}^{ac, J} (D_I Y_b^\beta) \partial_\beta^{b, I} Q_{c, J}^\gamma + \mathbb{H}_{\gamma\beta}^{cb, I} (D_I Y_b^\beta) \partial_\alpha^a Q_c^\gamma \right. \\ &\quad \left. + \frac{\partial P^j}{\partial x^j} \mathbb{H}_{\alpha\beta}^{ab, I} D_I Y_b^\beta \right) dA_a^\alpha \wedge \nu.\end{aligned}\tag{19}$$

Finally, a comparison of (16) with (19) yields equation (15), as required. \square

The following Lie derivative formula, as established in [1, 3], is central in the proof of our main Theorem.

Proposition 3. *Let T be a source form and X a projectable vector field on \mathcal{A} . Then*

$$\mathcal{L}_{\text{pr } X}T = \mathbb{E}(X_{\text{ev}} \lrcorner T) + \mathbb{H}_T(X_{\text{ev}}).\tag{20}$$

An extension of the Lie derivative formula (20) to non-projectable, generalized vector fields can be found in [1]. Note that for locally variational source forms T , equation (20) reduces to

$$\mathcal{L}_{\text{pr } X}T = \mathbb{E}(X_{\text{ev}} \lrcorner T).$$

Now

$$\mathcal{L}_{\text{pr } X} T = 0,$$

if the source form is invariant under X , or more precisely, if X is a distinguished symmetry of T , while

$$E(X_{\text{ev}} \lrcorner T) = 0,$$

if X generates a conservation law for T ; see [1]. Thus equation (20) provides a version of Noether's theorem for projectable vector fields expressed directly in terms of the system of differential equations without the explicit use of a Lagrangian.

On the other hand, in the situation of Takens' problem, each X belonging to the Lie algebra \mathfrak{g} of vector fields under consideration is assumed to be a distinguished symmetry of the source form T and to generate a conservation law for T , leading to the conditions $H_T(X_{\text{ev}}) = 0$ for all $X \in \mathfrak{g}$ for the Helmholtz operator of T . A basic objective in the analysis of Takens' problem is to identify on mathematical or physical grounds interesting classes \mathcal{T} of source forms (i.e. differential equations) and symmetry algebras \mathfrak{g} of vector fields so that one will be able to classify all \mathfrak{g} -invariant Helmholtz operators H_T corresponding to $T \in \mathcal{T}$ satisfying the conditions $H_T(X_{\text{ev}}) = 0$, $X \in \mathfrak{g}$.

3 Symmetries and conservation laws

In this section we analyze the relationship between symmetries [S1], [S2] and conservation laws [C1], [C2] in gauge field theories. As is well known, the commutation formula (11) and Noether's first and second theorems imply that a source form $T = E(\lambda)$ that is the Euler-Lagrange expression of a Lagrangian form λ with symmetries [S1], [S2], also possesses symmetries [S1], [S2] and, in addition, conservation laws [C1], [C2]. The following result, which is a generalization of those appearing in [14, 15], is a slight but non-vacuous extension of the above conclusions furnished by the classical Noether's theorems.

Proposition 4. *Suppose that the source form T is locally variational. Then T admits symmetries [S1] and [S2] if and only if it admits conservation laws [C1] and [C2].*

Proof. By assumption the Helmholtz operator H_T of T vanishes, and so equation (20) reduces to

$$\mathcal{L}_{\text{pr } X} T = E(X_{\text{ev}} \lrcorner T). \quad (21)$$

Recall that X generates a conservation law for T if and only if $E(X_{\text{ev}} \lrcorner T) = 0$. Thus the equivalence of [S1] and [C1] immediately follows from equation (21).

Next, with

$$X = Q_\varphi = (\varphi_{,a}^\alpha + c_{\beta\gamma}^\alpha A_a^\beta \varphi^\gamma) \frac{\partial}{\partial A_a^\alpha} \in \mathfrak{ga}(n),$$

equation (20) becomes

$$\mathcal{L}_{\text{pr } Q_\varphi} T = E(\varphi_{,a}^\alpha T_\alpha^a + c_{\beta\gamma}^\alpha A_a^\beta \varphi^\gamma T_\alpha^a).$$

We integrate by parts to write the right-hand side as

$$\mathbb{E}(\varphi_{,a}^\alpha T_\alpha^a + c_{\beta\gamma}^\alpha A_a^\beta \varphi^\gamma T_\alpha^a) = \mathbb{E}(-\varphi^\alpha D_a T_\alpha^a + c_{\beta\gamma}^\alpha A_a^\beta \varphi^\gamma T_\alpha^a) = -\mathbb{E}(\varphi^\alpha \nabla_a T_\alpha^a),$$

after which we have

$$\mathcal{L}_{\text{pr } Q_\varphi} T = -\mathbb{E}(\varphi^\alpha \nabla_a T_\alpha^a). \quad (22)$$

Equation (22) immediately shows that if T has conservation laws [C2] then it also has symmetries [S2]. It thus remains to prove that the condition

$$\mathbb{E}(\varphi^\alpha \nabla_a T_\alpha^a) = 0 \quad \text{for all } \varphi \in C^\infty(\mathbb{R}^n, \mathfrak{g}),$$

implies that $\nabla_a T_\alpha^a = 0$.

For this, suppose that for some α_o , $\nabla_a T_{\alpha_o}^a = F(x^i, A^{[l]})$ is of order l and that for some b, β and J with $|J| = l$,

$$(\partial_\beta^{b,J} F)(x_o^i, A_o^{[l]}) \neq 0, \quad (x_o^i, A_o^{[l]}) \in J^l(\mathcal{A}).$$

Now choose $\varphi_o \in C^\infty(\mathbb{R}^n, \mathfrak{g})$ such that

$$\frac{\partial^{|J|} \varphi_o^\alpha}{\partial x^J} (x_o^i) = \delta_{\alpha_o}^\alpha, \quad \frac{\partial^{|K|} \varphi_o^\alpha}{\partial x^K} (x_o^i) = 0, \quad K \neq J.$$

Then

$$\begin{aligned} \mathbb{E}_\beta^b(\varphi_o^\alpha (\nabla_a T_\alpha^a))(x_o^i, A_o^{[2l]}) &= \sum_{|I| \leq l} (-1)^{|I|} D_I \left(\varphi_o^\alpha \partial_\beta^{b,I} (\nabla_a T_\alpha^a) \right) (x_o^i, A_o^{[2l]}) \\ &= (\partial_\beta^{b,J} F)(x_o^i, A_o^{[l]}) \neq 0, \end{aligned}$$

which is a contradiction. Thus, inductively, we see that $\nabla_a T_\alpha^a = h_\alpha(x^i)$ are functions of x^i only. But due to the translational invariance of the source form T , each h_α must be constant. Finally, by the definition of the covariant divergence (3) and the total derivative operators (5), the covariant divergence $\nabla_a T_\alpha^a$ vanishes when each $A_{a,I}^\alpha = 0$, showing that $h_\alpha = 0$. QED

The next result shows that in gauge field theories, somewhat surprisingly, symmetries [S1], [S2] and conservation laws [C1], [C2] are not mutually independent.

Proposition 5. *Suppose that a source form T possesses symmetries [S1] and [S2] and conservation laws [C1]. Then T also admits conservation laws [C2].*

Proof. First, with $X = Q_\varphi = (\varphi_{,a}^\alpha + c_{\beta\gamma}^\alpha A_a^\beta \varphi^\gamma) \partial / \partial A_a^\alpha \in \mathfrak{ga}(n)$, equation (15) becomes

$$\sum_{|J| \geq 0} ((D_J Q_{\varphi_c}^\gamma) \partial_\gamma^{c,J} H_{\alpha\beta}^{ab,I} + (\partial_\beta^{b,I} D_J Q_{\varphi_c}^\gamma) H_{\alpha\gamma}^{ac,J}) + (\partial_\alpha^a Q_{\varphi_c}^\gamma) H_{\gamma\beta}^{cb,I} = 0. \quad (23)$$

The source form T has symmetries [S1] and conservation laws [C1] and so by equation (20),

$$\sum_{|I| \geq 0} A_{b,Ip}^\beta H_{\alpha\beta}^{ab,I} = 0, \quad p = 1, \dots, n. \quad (24)$$

Now apply the vector field $\text{pr } Q_\varphi$ to the above equation to see that

$$\sum_{|I| \geq 0} (D_{Ip} Q_{\varphi b}^\beta) H_{\alpha\beta}^{ab,I} + \sum_{|I|, |J| \geq 0} A_{b,Ip}^\beta (D_J Q_{\varphi c}^\gamma) (\partial_\gamma^{c,J} H_{\alpha\beta}^{ab,I}) = 0. \quad (25)$$

Combining (23) and (25) we obtain

$$\sum_{|I| \geq 0} (D_{Ip} Q_{\varphi b}^\beta) H_{\alpha\beta}^{ab,I} - \sum_{|I| \geq 0} A_{b,Ip}^\beta \left(\sum_{|J| \geq 0} (\partial_\beta^{b,I} D_J Q_{\varphi c}^\gamma) H_{\alpha\gamma}^{ac,J} + (\partial_\alpha^a Q_{\varphi c}^\gamma) H_{\gamma\beta}^{cb,I} \right) = 0,$$

which, on account of the definition of the total derivative operators (5) and equation (24), simplifies to

$$\sum_{|I| \geq 0} \left(\frac{\partial}{\partial x^p} D_I Q_{\varphi b}^\beta \right) H_{\alpha\beta}^{ab,I} = \sum_{|I| \geq 0} \left(D_I Q_{\frac{\partial \varphi}{\partial x^p} b}^\beta \right) H_{\alpha\beta}^{ab,I} = 0.$$

But the function $\varphi \in C^\infty(\mathbb{R}^n, \mathfrak{g})$ is arbitrary so we are able to conclude that

$$\sum_{|I| \geq 0} (D_I Q_{\varphi b}^\beta) H_{\alpha\beta}^{ab,I} = 0 \quad \text{for all } \varphi \in C^\infty(\mathbb{R}^n, \mathfrak{g}),$$

that is,

$$H_T(Q_\varphi) = 0 \quad \text{for all } \varphi \in C^\infty(\mathbb{R}^n, \mathfrak{g}). \quad (26)$$

Due to the gauge invariance of the source form T and equation (26), the Lie derivative formula (20) yields

$$E(\varphi_{,a}^\alpha T_\alpha^a + c_{\beta\gamma}^\alpha A_a^\beta \varphi^\gamma T_\alpha^a) = 0.$$

Now we continue as in the second part of the proof of Proposition 4 to conclude that $\nabla_a T_\alpha^a = 0$, as required. \square

4 Proof of Theorem 1

The proof of Theorem 1 relies on the following Lemma, which is a special case of a more general result presented in [2, 24].

Lemma 6. *Let $T = T_\alpha^a dA_a^\alpha \wedge \nu$, $T_\alpha^a = T_\alpha^a(x^i, A^{[k]})$, be a k -th order source form such that the covariant divergence $\nabla_a T_\alpha^a = 0$ vanishes identically. Then the component functions T_α^a are polynomials in the k th order derivative variables $A_{a,i_1 \dots i_k}^\alpha$ of degree at most $n - 1$.*

Proof. By assumption,

$$\frac{\partial T_\alpha^a}{\partial x^a} + \sum_{|I| \geq 0} A_{b,Ia}^\beta \partial_\beta^{b,I} T_\alpha^a + c_{\alpha\beta}^\gamma A_a^\beta T_\gamma^a = 0. \quad (27)$$

Now terms in (27) involving the order $k+1$ variables $A_{b,J}^\beta$, $|J| = k+1$, yield the equations

$$\partial_\beta^{b,(I} T_\alpha^a) = 0, \quad |I| = k. \quad (28)$$

Write $\partial_{\beta,k,X}^b$ for the partial differential operator

$$\partial_{\beta,k,X}^b = \sum_{|I|=k} X_I \partial_\beta^{b,I} = \sum_{1 \leq i_1, \dots, i_k \leq n} X_{i_1} \cdots X_{i_k} \partial_\beta^{b,i_1 \cdots i_k}, \quad (29)$$

where $X = (X_1, \dots, X_n) \in (\mathbb{R}^n)^*$ is a covector on \mathbb{R}^n . Then equation (28) is equivalent to

$$X_a \partial_{\beta,k,X}^b T_\alpha^a = 0 \quad \text{for all } X. \quad (30)$$

Next let $G_{\beta_1}^{b_1} \cdots G_{\beta_n \alpha}^{b_n}$ denote the mappings

$$G_{\beta_1}^{b_1} \cdots G_{\beta_n \alpha}^{b_n}(X^1, \dots, X^n, Y) = \partial_{\beta_1, k, X^1}^{b_1} \cdots \partial_{\beta_n, k, X^n}^{b_n} T_\alpha^a Y_a.$$

The operator T_α^a is polynomial in $A_{\alpha, I}^a$, $|I| = k$, of degree at most $n-1$ if and only if the mappings $G_{\beta_1}^{b_1} \cdots G_{\beta_n \alpha}^{b_n}$ vanish identically. But by (30) and multilinearity, the equation

$$G_{\beta_1}^{b_1} \cdots G_{\beta_n \alpha}^{b_n}(X^1, \dots, X^n, Y) = 0$$

holds whenever Y is a linear combination of the covectors X^1, \dots, X^n . Consequently $G_{\beta_1}^{b_1} \cdots G_{\beta_n \alpha}^{b_n}$ vanishes for almost all $X^1, \dots, X^n, Y \in (\mathbb{R}^n)^*$. By continuity, $G_{\beta_1}^{b_1} \cdots G_{\beta_n \alpha}^{b_n}$ must vanish identically. QED

Proof of Theorem 1. Part 2 of the Theorem is a straightforward consequence of theorem 2.1 in [4]. As to part 1, the source form T also admits conservation law [C2] by Proposition 5. Thus by letting $X = Q_\varphi = (\varphi_\alpha^\alpha + c_{\beta\gamma}^\alpha A_a^\beta \varphi^\gamma) \partial_\alpha^a$ in (20) we see that

$$\begin{aligned} & H_{\alpha\beta}^{ij,kl} \varphi_{,jkl}^\beta + (H_{\alpha\beta}^{ij,k} + c_{\zeta\beta}^\gamma A_l^\zeta H_{\alpha\gamma}^{il,jk}) \varphi_{,jk}^\beta \\ & + (H_{\alpha\beta}^{ij} + c_{\zeta\beta}^\gamma (A_k^\zeta H_{\alpha\gamma}^{ik,j} + 2A_{k,l}^\zeta H_{\alpha\gamma}^{ik,jl})) \varphi_{,j}^\beta \\ & + c_{\zeta\beta}^\gamma (A_j^\zeta H_{\alpha\gamma}^{ij} + A_{j,k}^\zeta H_{\alpha\gamma}^{ij,k} + A_{j,kl}^\zeta H_{\alpha\gamma}^{ij,kl}) \varphi^\beta = 0. \end{aligned}$$

In particular, we have that

$$H_{\alpha\beta}^{ij} = c_{\beta\zeta}^\gamma (A_k^\zeta H_{\alpha\gamma}^{ik,j} + 2A_{k,l}^\zeta H_{\alpha\gamma}^{ik,jl}), \quad (31)$$

so that it suffices to show that the first and second order Helmholtz conditions of T vanish in order to prove that the source form T is locally variational.

Next, with $X = \partial/\partial x^p$, equation (20) becomes

$$A_{j,p}^\beta H_{\alpha\beta}^{ij} + A_{j,kp}^\beta H_{\alpha\beta}^{ij,k} + A_{j,klp}^\beta H_{\alpha\beta}^{ij,kl} = 0,$$

which, together with (31), yields the equation

$$(A_{j,kp}^\gamma + c_{\beta\zeta}^\gamma A_j^\zeta A_{k,p}^\beta) H_{\alpha\gamma}^{ij,k} + (A_{j,klp}^\gamma + 2c_{\beta\zeta}^\gamma A_{k,p}^\beta A_{j,l}^\zeta) H_{\alpha\gamma}^{ij,kl} = 0. \quad (32)$$

By Lemma 6, the components T_α^a of T are polynomial in the second order variables $A_{a,ij}^\alpha$ of degree $d \leq n - 1$. Thus, by virtue of the expressions (14), the first order components $H_{\alpha\beta}^{ab,i}$ are linear in the third order derivative variables with coefficients that are polynomial in $A_{a,ij}^\alpha$ of degree at most $d - 2$, and the remaining terms in $H_{\alpha\beta}^{ab,i}$ are polynomial in $A_{a,ij}^\alpha$ of degree at most d . Moreover, the second order components $H_{\alpha\beta}^{ab,ij}$ are of degree at most $d - 1$ in the variables $A_{a,ij}^\alpha$. So, in particular, when $n = 2$, the first and second order components $H_{\alpha\beta}^{ab,i}$, $H_{\alpha\beta}^{ab,ij}$ are functions of $A^{[2]}$, $A^{[1]}$ only, respectively.

Next assume that $n \geq 3$. Apply the operator $\partial_{\eta,3,X}^a$ as in (29) to equation (32) to see that

$$(A_{j,kp}^\gamma + c_{\beta\zeta}^\gamma A_j^\zeta A_{k,p}^\beta) \partial_{\eta,3,X}^a H_{\alpha\gamma}^{ij,k} + X_p X_k X_l H_{\alpha\eta}^{ia,kl} = 0. \quad (33)$$

Denote the degrees of $\partial_{\alpha,3,X}^a H_{\eta\gamma}^{ij,k}$, $H_{\alpha\eta}^{ia,kl}$ in the variables $A_{a,ij}^\alpha$ by d_1 , d_2 , respectively. It now follows from (33) that $d_1 \leq d_2 - 1$. In fact, if $d_1 \geq d_2$, then an application of the differential operator

$$\partial_{\beta_1,2,Y^1}^{b_1} \cdots \partial_{\beta_{d_1+1},2,Y^{d_1+1}}^{b_{d_1+1}}$$

to (33) yields the equation

$$\sum_{s=1}^{d_1+1} Y_p^s B^s = 0, \quad p = 1, \dots, n, \quad (34)$$

where

$$B^s = Y_k^s \partial_{\beta_1,2,Y^1}^{b_1} \cdots \widehat{\partial_{\beta_s,2,Y^s}^{b_s}} \cdots \partial_{\beta_{d_1+1},2,Y^{d_1+1}}^{b_{d_1+1}} \partial_{\eta,3,X}^a H_{\alpha\beta_s}^{ib_s,k}. \quad (35)$$

Since $d_1 \leq n - 3$, it follows from (34) that the expressions B^s vanish when the covectors Y^1, \dots, Y^{d_1+1} are linearly independent, and thus by continuity, they vanish identically, which yields a contradiction.

Hence we can assume that $d_1 \leq d_2 - 1$, and so, in particular, $d_2 \geq 1$. Now apply the d_2 -fold differential operator $\partial_{\beta_1,2,Y^1}^{b_1} \cdots \partial_{\beta_{d_2},2,Y^{d_2}}^{b_{d_2}}$ to (33) to derive the expression

$$\sum_{s=1}^{d_2} Y_p^s B^s + X_p N = 0, \quad p = 1, \dots, n, \quad (36)$$

where

$$B^s = Y_k^s \partial_{\beta_1,2,Y^1}^{b_1} \cdots \widehat{\partial_{\beta_s,2,Y^s}^{b_s}} \cdots \partial_{\beta_{d_2},2,Y^{d_2}}^{b_{d_2}} \partial_{\eta,3,X}^a H_{\alpha\beta_s}^{ib_s,k},$$

$$N = X_k X_l \partial_{\beta_1,2,Y^1}^{b_1} \cdots \partial_{\beta_{d_2},2,Y^{d_2}}^{b_{d_2}} H_{\alpha\eta}^{ia,kl}.$$

As above, equation (36) implies that $B^s = 0$, $N = 0$, which again is a contradiction, and we deduce that the second order components $H_{\alpha\beta}^{ab,ij}$ and the derivative terms $\partial_{\eta,3,X}^a H_{\alpha\gamma}^{ij,k}$ are functions of $A^{[1]}$ only. But then, once more with the help of (33), we are able to conclude that in all the cases $n \geq 2$, the second order Helmholtz conditions $H_{\alpha\beta}^{ab,ij} = 0$ are satisfied and that the first order components $H_{\alpha\beta}^{ab,i}$ must be functions of $A^{[2]}$ only.

With this, equation (32) reduces to

$$(A_{j,kp}^\gamma + c_{\beta\zeta}^\gamma A_j^\zeta A_{k,p}^\beta) H_{\alpha\gamma}^{ij,k} = 0,$$

where, by the above, the components $H_{\alpha\gamma}^{ij,k} = H_{\alpha\gamma}^{ij,k}(A^{[2]})$ are polynomial of degree $d \leq n-1$ in the second order variables $A_{a,ij}^\alpha$. Now one can easily repeat the above arguments to show that the first order Helmholtz conditions $H_{\alpha\beta}^{ab,i} = 0$ are also satisfied. Consequently, the source form T is locally variational. \square QED

5 Examples

We let $\kappa_{\alpha\beta} = c_{\alpha\zeta}^\gamma c_{\beta\gamma}^\zeta$ denote the components of the Killing form κ of the Lie algebra \mathfrak{g} in a given basis $\{e_\alpha\}$ and we will use $\kappa_{\alpha\beta}$ to lower Lie algebra indices in what follows. The components of the curvature, or field strength, of the gauge field A_a^α are given by

$$f_{ab}^\alpha = A_{b,a}^\alpha - A_{a,b}^\alpha + c_{\beta\gamma}^\alpha A_a^\beta A_b^\gamma. \quad (37)$$

In coordinate expressions a semicolon will indicate a covariant derivative so that, for example, $f_{ab;i}^\alpha = D_i f_{ab}^\alpha + c_{\beta\gamma}^\alpha A_i^\beta f_{ab}^\gamma$. Furthermore, we raise and lower the underlying spatial indices using the Minkowski metric $\eta = \text{diag}(-1, 1, \dots, 1)$.

In analogy with the construction presented in [5], section 4, given a differential operator

$$S: J^k(\mathcal{A}) \rightarrow \Lambda^2(T\mathbb{R}^n) \otimes \mathfrak{g}^* \quad (38)$$

with components $S_\alpha^{ab} = S_\alpha^{[ab]}(A^{[k]})$, we associate to it a source form T of order $k+1$ on \mathcal{A} with the components

$$T_\alpha^a = \nabla_b S_\alpha^{ab} = D_b S_\alpha^{ab} + c_{\alpha\beta}^\gamma A_b^\beta S_\gamma^{ab}. \quad (39)$$

Lemma 7. *Suppose that the differential operator S has symmetries [S1] and that its components S_α^{ab} satisfy the conditions*

$$c_{\alpha\gamma}^\beta f_{ab}^\gamma S_\beta^{ab} = 0, \quad \alpha = 1, \dots, r, \quad (40)$$

$$f_{ab;p}^\alpha S_\alpha^{ab} = D_i s_p^i, \quad p = 1, \dots, n, \quad (41)$$

$$\mathcal{L}_{\text{pr } Q_\varphi} S_\alpha^{ab} = -c_{\alpha\gamma}^\beta S_\beta^{ab} \varphi^\gamma, \quad Q_\varphi \in \mathfrak{ga}(n), \quad (42)$$

where the s_p^i are some differential functions. Then the source form T with components T_α^a as in (39) admits symmetries [S1], [S2] and conservation laws [C1], [C2].

Proof. The source form T clearly admits translational symmetries [S1]. We compute

$$\nabla_a T_\alpha^a = \nabla_a \nabla_b S_\alpha^{ab} = \nabla_{[a} \nabla_{b]} S_\alpha^{ab} = \frac{1}{2} c_{\alpha\gamma}^\beta f_{ab}^\gamma S_\beta^{ab},$$

from which we see that T admits conservation laws [C2] exactly when equation (40) holds true.

Next note that

$$f_{ab;p}^\alpha S_\alpha^{ab} = (D_p f_{ab}^\alpha + c_{\beta\gamma}^\alpha A_p^\beta f_{ab}^\gamma) S_\alpha^{ab},$$

so, on account of (40), equation (41) holds if and only if

$$(D_p f_{ab}^\alpha) S_\alpha^{ab} = D_i s_p^i, \quad p = 1, \dots, n.$$

We compute

$$\begin{aligned} (D_p f_{ab}^\alpha) S_\alpha^{ab} &= 2(-D_b D_p A_a^\alpha + c_{\beta\gamma}^\alpha A_{a,p}^\beta A_b^\gamma) S_\alpha^{ab} \\ &= D_b(-2A_{a,p}^\alpha S_\alpha^{ab}) + 2A_{a,p}^\alpha (D_b S_\alpha^{ab} + c_{\alpha\gamma}^\beta A_b^\gamma S_\beta^{ab}) \\ &= D_b(-2A_{a,p}^\alpha S_\alpha^{ab}) + 2A_{a,p}^\alpha T_\alpha^a, \end{aligned}$$

where we first used (37) and the skew-symmetry of S_α^{ab} in the index pair a, b to derive the expression on the right-hand side of the first line of the equation, and then we integrated by parts to obtain the expression on the second line. Thus if equations (40), (41) hold, then

$$A_{a,p}^\alpha T_\alpha^a = D_b(A_{a,p}^\alpha S_\alpha^{ab}) + \frac{1}{2} D_i s_p^i, \quad (43)$$

and hence T admits conservation laws [C1] with $t_p^i = A_{a,p}^\alpha S_\alpha^{ai} + \frac{1}{2} s_p^i$.

Finally, we recall the general fact that given a \mathfrak{g} -valued differential function G^α transforming homogeneously under gauge transformations then its covariant derivatives $\nabla_i G^\alpha = D_i G^\alpha + c_{\beta\gamma}^\alpha A_i^\beta G^\gamma$ also transform homogeneously. Thus, on account of (42), the source form T admits symmetries [S2]. \square

Example 8. In this example we derive translationally invariant non-trivial second order solutions to equations (40), (41), (42), when $n \geq 3$. These, via (38), will furnish examples of third order source forms that admit symmetries [S1], [S2] and conservation laws [C1], [C2] but are not locally variational, showing that Theorem 1 is sharp as regards the order of the source form.

First, due to the skew-symmetry of the constants $c_{\alpha\beta\gamma} = \kappa_{\alpha\nu} c_{\beta\gamma}^\nu$ in α, β, γ , one immediately sees that for any functions $q^{ab} = q^{(ab)}(A^{[k]})$ defined on some $J^k(\mathcal{A})$, $k \geq 0$, the products

$$S_\alpha^{ab} = q^{ab} f_\alpha^{ab} \quad (\text{no summation in } a, b) \quad (44)$$

satisfy equations (40).

Next, for definiteness, assume that the functions s_p^i in (41) are constant coefficient linear expressions in the independent variables x^i so that equation (41) reduces to

$$f_{ab;p}^\alpha S_\alpha^{ab} = \lambda_p, \quad p = 1, \dots, n, \quad (45)$$

where λ_p are some constants. Write $m = n(n+1)/2$ for the dimension of the symmetric product $S^2(T^*\mathbb{R}^n)$. Substitution of expressions (44) into (45) yields an $n \times m$ system of linear equations

$$h_{p,ab} q^{ab} = \lambda_p, \quad p = 1, \dots, n, \quad (46)$$

for the differential functions q^{ab} with gauge invariant coefficients

$$h_{p,ab} = \sum_{\alpha=1}^r f_{\alpha}^{ab} f_{ab;p}^{\alpha} \quad (\text{no summation in } a, b),$$

that are symmetric in the index pair a, b . Thus, in particular, the invariance condition (42) is automatically satisfied for S_{α}^{ab} in (44) when the q^{ab} are constructed as functions in the coefficients $h_{p,ab}$ only.

Next let $\mathcal{E}^{a_1 b_1 \dots a_m b_m}$ denote a non-zero rank $2m$ constant tensor on \mathbb{R}^n with the symmetries

$$\begin{aligned} \mathcal{E}^{a_1 b_1 \dots b_i a_i \dots a_m b_m} &= \mathcal{E}^{a_1 b_1 \dots a_i b_i \dots a_m b_m}, & \text{and} \\ \mathcal{E}^{a_1 b_1 \dots a_j b_j \dots a_i b_i \dots a_m b_m} &= -\mathcal{E}^{a_1 b_1 \dots a_i b_i \dots a_j b_j \dots a_m b_m}. \end{aligned} \quad (47)$$

One can regard $\mathcal{E}^{a_1 b_1 \dots a_m b_m}$ as the permutation symbol on $S^2(T^*\mathbb{R}^n)$.

In the case $n = m = 3$, a solution to (46) is furnished by

$$q^{ab} = V^{-1} V^{p,ab} \lambda_p, \quad (48)$$

where

$$\begin{aligned} V &= \frac{1}{6} \epsilon^{p_1 p_2 p_3} \mathcal{E}^{a_1 b_1 a_2 b_2 a_3 b_3} h_{p_1, a_1 b_1} h_{p_2, a_2 b_2} h_{p_3, a_3 b_3}, & \text{and} \\ V^{p,ab} &= \frac{1}{2} \epsilon^{pp_1 p_2} \mathcal{E}^{aba_1 b_1 a_2 b_2} h_{p_1, a_1 b_1} h_{p_2, a_2 b_2}, \end{aligned}$$

and where $\epsilon^{p_1 p_2 p_3}$ denotes the usual 3-dimensional permutation symbol.

If $n \geq 4$, we set $\lambda_p = 0$, $p = 1, \dots, n$, in which case solutions to (46) can be constructed from the $n \times n$ minors of the matrix $(h_{p,ab})$ using the standard formula

$$q^{ab} = \mathcal{E}^{aba_1 b_1 \dots a_n b_n a_{n+1} b_{n+1} \dots a_{m-1} b_{m-1}} \tau_{a_{n+1} b_{n+1} \dots a_{m-1} b_{m-1}} h_{1, a_1 b_1} \dots h_{n, a_n b_n}, \quad (49)$$

where we let the coefficient functions $\tau_{a_{n+1} b_{n+1} \dots a_{m-1} b_{m-1}} = \tau_{a_{n+1} b_{n+1} \dots a_{m-1} b_{m-1}}(h_{p,ab})$ depend on the gauge invariant quantities $h_{p,ab}$ and have the same symmetries in their indices as $\mathcal{E}^{a_1 b_1 \dots a_m b_m}$.

By the above, with q^{ab} as in (48), (49), the components S_{α}^{ab} in (44) satisfy conditions (40), (41), (42) of Lemma 7, so it only remains to show that the source form T constructed from S fails to be locally variational. To this end we analyze the third order components

$$H_{\alpha\beta}^{11,222} = \partial_{\beta}^{1,222} T_{\alpha}^1 + \partial_{\alpha}^{1,222} T_{\beta}^1$$

of the Helmholtz operator H_T of T . Keeping in mind that the order of S_α^{ab} in the coordinates (3) is two, we compute

$$\partial_\beta^{1,222} T_\alpha^1 = \partial_\beta^{1,222} \nabla_c S_\alpha^{1c} = \partial_\beta^{1,22} S_\alpha^{12} = (\partial_\beta^{1,22} q^{12}) f_\alpha^{12},$$

which, with the help of (48), (49), yields

$$H_{\alpha\beta}^{11,222} = 4V^{-2} V^{2,12} V^{p,12} \lambda_p f_\alpha^{12} f_\beta^{12}, \quad \text{when } n = 3, \quad (50)$$

and

$$H_{\alpha\beta}^{11,222} = -2\mathcal{E}^{12a_1b_1 \dots a_n b_n a_{n+1} b_{n+1} \dots a_{m-1} b_{m-1}} \times \left(\frac{\partial}{\partial h_{2,12}} \tau_{a_{n+1} b_{n+1} \dots a_{m-1} b_{m-1}} \right) h_{1,a_1 b_1} \dots h_{n,a_n b_n} f_\alpha^{12} f_\beta^{12}, \quad \text{when } n \geq 4. \quad (51)$$

If the Killing form κ of \mathfrak{g} is non-trivial, so in particular when \mathfrak{g} is semi-simple, we see from (50), (51) that in general, the Helmholtz operator H_T of T is non-vanishing and that, consequently, part 1 of Theorem 1 does not extend to third order source forms when $n \geq 3$. Additionally, equation (51) with $\tau_{a_{n+1} b_{n+1} \dots a_{m-1} b_{m-1}}$ chosen to be a suitable polynomial in $h_{p,ab}$ shows that when $n \geq 4$, part 2 of Theorem 1 fails for polynomial source forms of degree $d \geq n + 2$ in the variables $A_{a,I}^\alpha$.

Remark 9. If the Killing form κ of \mathfrak{g} vanishes identically, then, in particular, \mathfrak{g} must be solvable, and, when $n \geq 3$, examples presented in [4] can be straightforwardly adapted to provide instances of third order non-variational source forms with symmetries [S1], [S2] and conservation laws [C1], [C2]. We will omit the details in the interest of brevity.

6 Conclusions

In this paper we prove that a system of second order gauge field equations admitting translational and gauge symmetries and the associated conservation laws can be written as the Euler-Lagrange expression of some Lagrangian function. We also show that our result is sharp when the underlying space is at least 3 dimensional by constructing explicit examples of non-variational third-order systems with the required symmetries and conservation laws. However, the optimal form of Theorem 1 for gauge fields in 2 dimensions, which include Yang-Mills fields on Riemann surfaces [6] and the physically important theory of vortices [16], remains an open problem.

A worthwhile generalization of the present work would be to extend the symmetry group to include Lorentz transformations or conformal transformations of the underlying Minkowski space. However, results in [4] intimate that with the extended symmetry group, analysis of the associated Helmholtz conditions, with the source form T now being of third order, will become exceedingly intricate. Additionally, it would be a significant problem to determine whether the Lagrangian for a source form, whose existence is guaranteed by Theorem 1, can be chosen to be invariant under the given symmetry

group. This question is tantamount to solving the invariant inverse problem of calculus of variations for Yang-Mills fields with the infinite dimensional pseudo-group generated by translations and gauge transformations. Preliminary results in this direction can be found in [27], see also [18].

References

- [1] I. M. Anderson, The Variational Bicomplex, Utah State University Technical Report, 1989; http://www.math.usu.edu/~fg_mp
- [2] I. M. Anderson, T. Duchamp, *On the existence of global variational principles*, Amer. J. Math. **102** (1980), 781–868.
- [3] I. M. Anderson, J. Pohjanpelto, *Variational principles for differential equations with symmetries and conservation laws I: Second order scalar equations*, Math. Ann. **299** (1994), 191–222.
- [4] I. M. Anderson, J. Pohjanpelto, *Variational principles for differential equations with symmetries and conservation laws II: Polynomial differential equations*, Math. Ann. **301** (1995), 627–653.
- [5] I. M. Anderson, J. Pohjanpelto, *Symmetries, conservation laws and variational principles for vector field theories*, Math. Proc. Camb. Philos. Soc. **120** (1996), 369–384.
- [6] M. F. Atiyah, R. Bott, *The Yang-Mills Equations over Riemann Surfaces*, Philos. Trans. R. Soc. Lond. Ser. A **308** (1983), 523–615.
- [7] E. Cartan, *Sur les équations de la gravitation d'Einstein*, J. Math. Pure Appl. **1** (1922), 141–204.
- [8] S. K. Donaldson, P. B. Kronheimer, *The geometry of four-manifolds*, Oxford University Press, New York, 1990.
- [9] F. Etayo Gordejuela, P. L. García Pérez, J. Muñoz Masqué, J. Pérez Alvarez, *Higher-order Utiyama-Yang-Mills Lagrangians* J. Geom. Phys. **57** (2007), 1089–1097.
- [10] P. H. Frampton, *Gauge Field Theories*, 2nd ed., Wiley, New York, 2000.
- [11] A. Fuster, M. Henneaux, A. Maas, *BRST Quantization: a Short Review*, Int. J. Geom. Methods Mod. Phys. **2** (2005), 939–963.
- [12] D. M. Gitman, I. V. Tyutin, *Quantization of fields with constraints*, Springer, New York, 1990.

- [13] G. W. Horndeski, *Differential operators associated with the Euler-Lagrange operator*, Tensor **28** (1974), 303–318.
- [14] G. W. Horndeski, *Gauge invariance and charge conservation*, Tensor **32** (1978), 131–139.
- [15] G. W. Horndeski, *Gauge invariance and charge conservation in non-Abelian gauge theories*, Arch. Rat. Mech. Anal. **75** (1981), 211–227.
- [16] A. Jaffe, C. Taubes, *Vortices and monopoles*, Progress in Physics **2**, Birkhuser, Mass. (1980).
- [17] J. Janyška, *Higher-order Utiyama invariant interaction*, Rep. Math. Phys. **59** (2007), 63–81.
- [18] M. C. López, R. J. Noriega, C. G. Schifini, *The equivariant inverse problem and the uniqueness of the Yang-Mills equations*, J. Math. Phys. **30** (1989), 2382–2387.
- [19] D. Lovelock, *The Einstein tensor and its generalizations*, J. Math. Phys. **12** (1971), 498–501.
- [20] D. Lovelock, *Vector-tensor field theories and the Einstein-Maxwell field equations*, Proc. Roy. Soc. London Ser. A **341** (1974), 285–297.
- [21] D. Lovelock, *Divergence-free third order concomitants of the metric tensor in three dimensions*, Topics in differential geometry (in memory of Evan Tom Davies), pp. 87–98. Academic Press, New York, 1976.
- [22] D. Lovelock, *Bivector field theories, divergence-free vectors and the Einstein-Maxwell field equations*, J. Math. Phys. **18** (1977), 1491–1498.
- [23] J. M. Morgan, *The Seiberg-Witten Equations and Applications to the Topology of Smooth Four-Manifolds*, Princeton University Press, Princeton, 1995.
- [24] P. J. Olver, *Conservation laws and null divergences*, Math. Proc. Camb. Philos. Soc. **94** (1983), 529–540.
- [25] P. J. Olver, *Applications of Lie Groups to Differential Equations*, GTM **107**, 2nd ed., Springer, New York, 1993.
- [26] J. Pohjanpelto, *Takens' problem for systems of first order differential equations*, Ark. Mat. **33** (1995), 343–356.
- [27] J. Pohjanpelto, I. M. Anderson, *Infinite dimensional Lie algebra cohomology and the cohomology of invariant Euler-Lagrange complexes: a preliminary report*, Proceedings of the 6th International Conference on Differential Geometry and Applications, Brno, 1995 (J. Janyška, I. Kolar, J. Slovák, eds.) Masaryk University, Brno, Czech Republic 1996, pp. 427–448.

- [28] F. Takens, *Symmetries, conservation laws and variational principles*, Lecture Notes in Mathematics No. 597, Springer, New York (1977), pp. 581–603.
- [29] H. Vermeil, *Notiz über das mittlere Krümmungsmass einer n -fach Riemannschen Mannigfaltigkeit*, Akad. Wiss. Göttingen Nachr. (1917), 334–344.
- [30] H. Weyl, *Space-Time-Matter*, 4th ed., Dover, New York, 1922.