MOVING FRAMES AND DIFFERENTIAL INVARIANTS FOR LIE PSEUDO-GROUPS

Peter Olver

School of Mathematics, University of Minnesota,
Minneapolis, MN 55455, USA
E-mail: olver@math.umn.edu
http://www.math.umn.edu/∼olver

Juha Pohjanpelto

Department of Mathematics, Oregon State University,
Corvallis, OR 97331, USA
E-mail: juha@math.oregonstate.edu
http://oregonstate.edu/∼pohjanpp

We survey a recent extension of the moving frames method for infinite-dimensional Lie pseudo-groups. Applications include a new, direct approach to the construction of Maurer–Cartan forms and their structure equations for pseudo-groups, and new algorithms, based on constructive commutative algebra, for uncovering the structure of the algebra of differential invariants for pseudo-group actions.

Keywords: pseudo-group; moving frame; differential invariant, Gröbner basis.

1. Introduction

Our goal in this contribution is to provide a brief survey of the moving frame theory for general Lie pseudo-groups recently put forth by the authors in Refs. 8–11, and in Refs. 3,4 in collaboration with J. Cheh. The moving frame construction is based on the interplay between two jet bundles: the infinite jets $\mathcal{D}^{(\infty)} \subset \mathrm{J}^\infty(M, M)$ of local diffeomorphisms and the infinite jets $\mathrm{J}^\infty(M, p)$ of p-dimensional submanifolds of M. Importantly, the invariant contact forms on $\mathcal{D}^{(\infty)}$ will play the role of Maurer–Cartan forms for the diffeomorphism pseudo-group and restricting these to the pseudo-group subbundle $\mathcal{G}^{(\infty)} \subset \mathcal{D}^{(\infty)}$ yields a complete system of Maurer–Cartan forms for the pseudo-group. Remarkably, the restricted Maurer–Cartan forms satisfy an "invariantized" version of the infinitesimal determining equations for the pseudo-group, which can immediately be used to produce an explicit form of the pseudo-group structure equations.

The construction of a moving frame is based on a choice of local cross-section to the pseudo-group orbits in $\mathrm{J}^n(M,p)$. The moving frame induces an invariantization process that projects general differential functions and differential forms on $\mathrm{J}^n(M,p)$ to their invariant counterparts yielding complete local systems of differential invariants and invariant coframes on $\mathrm{J}^n(M,p)$. The corresponding invariant total derivative operators will map invariants to invariants of higher order. The structure of the algebra of differential invariants, including the specification of a finite generating set of differential invariants along with their syzygies or differential relations, will then follow from the recurrence formulae that relate the differentiated and normalized differential invariants. It is worth emphasizing that this final step requires only linear algebra and differentiation based on the infinitesimal determining equations of the pseudo-group action. Except possibly for some low order complications, the structure of the differential invariant algebra is then governed by two commutative algebraic modules: the symbol module of the infinitesimal determining system of the pseudo-group and a new module, named the "prolonged symbol module", containing the symbols of the prolonged infinitesimal generators.

2. The Diffeomorphism Pseudo–Group

Let $\mathcal{D} = \mathcal{D}(M)$ denote the pseudo-group of all local diffeomorphisms of a smooth m-dimensional manifold M and, for each $0 \leq n \leq \infty$, let $\mathcal{D}^{(n)} \subset \mathrm{J}^n(M,M)$ stand for the n^{th} order diffeomorphism jet bundle. Local coordinates $(z,Z) = (z^1,\ldots,z^m,Z^1,\ldots,Z^m)$ on $M \times M$ induce local coordinates $(z,Z^{(n)})$ for $g^{(n)} = \mathrm{j}_n\varphi|_z \in \mathcal{D}^{(n)}$, where the components Z^a_B of $Z^{(n)}$ represent the partial derivatives $\partial^B\varphi^a/\partial z^B$ of φ at z.

The cotangent bundle $T^*\mathcal{D}^{(\infty)}$ splits into horizontal and vertical (or contact) components, cf. Refs. 1,7, with an induced splitting $d = d_M + d_G$ of the exterior derivative. In local coordinates $g^{(\infty)} = (z, Z^{(\infty)})$, the horizontal subbundle is spanned by the one-forms $dz^a = d_M\, z^a$, $a = 1,\ldots,m$, while the vertical subbundle is spanned by the basic *contact forms*

$$\Upsilon^a_B = d_G\, Z^a_B = dZ^a_B - \sum_{c=1}^{m} Z^a_{B,c}\, dz^c, \quad a = 1,\ldots,m, \quad \#B \geq 0. \qquad (1)$$

Composition of local diffeomorphisms induces an action of $\psi \in \mathcal{D}$ by right multiplication on diffeomorphism jets: $\mathrm{R}_\psi(\mathrm{j}_n\varphi|_z) = \mathrm{j}_n(\varphi\circ\psi^{-1})|_{\psi(z)}$.

A differential form μ on $\mathcal{D}^{(n)}$ is *right-invariant* if $\mathrm{R}_\psi^* \mu = \mu$, where defined, for every $\psi \in \mathcal{D}$.

The horizontal derivatives $\sigma^a = d_M Z^a = \sum_{b=1}^m Z_b^a \, dz^b$, $a = 1, \ldots, m$, of the right-invariant coordinate functions $Z^a \colon \mathcal{D}^{(0)} \to \mathbb{R}$ form an invariant horizontal coframe, while their vertical derivatives $\mu^a = d_G Z^a = \Upsilon^a$ are the zero$^{\text{th}}$ order invariant contact forms. Let $\mathbb{D}_{Z^1}, \ldots, \mathbb{D}_{Z^m}$ be the total derivative operators dual to the horizontal forms σ^a. Then the higher-order invariant contact forms are obtained by successively Lie differentiating the invariant contact forms μ^a,

$$\mu_B^a = \mathbb{D}_Z^B \mu^a = \mathbb{D}_Z^B \Upsilon^a, \quad a = 1, \ldots, m, \quad k = \# B \geq 0, \tag{2}$$

where $\mathbb{D}_Z^B = \mathbb{D}_{Z^{b_1}} \cdots \mathbb{D}_{Z^{b_k}}$. We view the right-invariant contact forms $\mu^{(\infty)} = (\, \ldots \, \mu_B^a \, \ldots \,)$ as the *Maurer–Cartan forms* for the diffeomorphism pseudo-group.

Next let $\mu[\![\, H \,]\!]$ denote the column vector whose components are the invariant contact form-valued formal power series

$$\mu^a[\![\, H \,]\!] = \sum_{\# B \geq 0} \frac{1}{B!} \, \mu_B^a \, H^B, \qquad a = 1, \ldots, m,$$

depending on the parameters $H = (H^1, \ldots, H^m)$. Further, let $dZ = \mu[\![\, 0 \,]\!] + \sigma$ denote the column vector of one-forms with entries $dZ^a = \mu^a + \sigma^a$.

Theorem 2.1. *The complete structure equations for the diffeomorphism pseudo-group are given by the power series identity*

$$d\mu[\![\, H \,]\!] = \nabla_H \mu[\![\, H \,]\!] \wedge \big(\mu[\![\, H \,]\!] - dZ \big), \qquad d\,\sigma = \nabla_H \mu[\![\, 0 \,]\!] \wedge \sigma, \tag{3}$$

where $\nabla_H \mu[\![\, H \,]\!] = \left(\dfrac{\partial \mu^a}{\partial H^b} [\![\, H \,]\!] \right)$ *denotes the Jacobian matrix.*

3. Lie Pseudo–Groups

The literature contains several variants of the precise technical definition of a Lie pseudo-group, see e.g. Refs. 2,5,6,12,13. Ours is:

Definition 1. A sub-pseudo-group $\mathcal{G} \subset \mathcal{D}$ will be called a *Lie pseudo-group* if there exists $n_0 \geq 1$ such that for all finite $n \geq n_0$:

(a) the corresponding sub-groupoid $\mathcal{G}^{(n)} \subset \mathcal{D}^{(n)}$ forms a smooth, embedded subbundle,
(b) every smooth local solution $Z = \varphi(z)$ to the determining system $\mathcal{G}^{(n)}$ belongs to $\mathcal{G}$,

(c) $\mathcal{G}^{(n)} = \mathrm{pr}^{(n-n_0)}\, \mathcal{G}^{(n_0)}$ is obtained by prolongation.

Let $\mathfrak{g} \subset \mathcal{X}$ denote the (local) Lie algebra of infinitesimal generators of the pseudo-group, i.e., the set of locally defined vector fields whose flows belong to $\mathcal{G}$. In local coordinates, we can view $\mathrm{J}^n\mathfrak{g} \subset \mathrm{J}^n TM$ as defining a formally integrable linear system of partial differential equations

$$L^{(n)}(z, \zeta^{(n)}) = 0 \tag{4}$$

for the vector field coefficients $\mathbf{v} = \sum_{a=1}^m \zeta^a(z)\partial/\partial z^a$, called the *linearized* or *infinitesimal determining equations* for the pseudo-group.

A complete system of right-invariant contact forms on $\mathcal{G}^{(\infty)} \subset \mathcal{D}^{(\infty)}$ is obtained by restricting the Maurer–Cartan forms (2) to $\mathcal{G}^{(\infty)}$. Remarkably, constraints among the restricted forms can be explicitly characterized by an invariant version of the linearized determining equations (4).

Theorem 3.1. *The linear system*

$$L^{(n)}(Z, \mu^{(n)}) = 0 \tag{5}$$

provides a complete set of dependencies among the Maurer–Cartan forms $\mu^{(n)}$ *on* $\mathcal{G}^{(n)}$. *The structure equations for the pseudo-group* $\mathcal{G}$ *are obtained by imposing (5) on the diffeomorphism structure equations (3).*

4. Pseudo–Group Actions on Extended Jet Bundles

For $0 \le n \le \infty$, let $\mathrm{J}^n = \mathrm{J}^n(M,p)$ denote the bundle of n^{th} order jets of p-dimensional submanifolds of M, cf. Ref. 7. We employ the standard local coordinates $z^{(n)} = (x, u^{(n)}) = (\ \ldots\ x^i\ \ldots\ u^\alpha_J\ \ldots\ )$ on J^n induced by a splitting of the local coordinates $z = (x,u) = (x^1,\ldots,x^p,u^1,\ldots,u^q)$ on M into p independent and $q = m-p$ dependent variables, Ref. 7. The pseudogroup $\mathcal{D}$ and their jets $\mathcal{D}^{(n)}$ act on J^n in an obvious fashion.

Let $\mathcal{H}^{(n)}$ denote the bundle obtained by pulling back the pseudo-group jet bundle $\mathcal{G}^{(n)} \to M$ via the projection $\widetilde{\pi}^n_0 \colon \mathrm{J}^n \to M$. Local coordinates on $\mathcal{H}^{(n)}$ are given by $(x, u^{(n)}, g^{(n)})$, where the coordinates $g^{(n)}$ parametrize the pseudo-group jets.

Definition 4.1. A *moving frame* $\rho^{(n)}$ of *order* n is a $\mathcal{G}^{(n)}$ equivariant local section of the bundle $\mathcal{H}^{(n)} \to \mathrm{J}^n$.

Thus, in local coordinates, a moving frame defines a right equivariant map $g^{(n)} = \gamma^{(n)}(x, u^{(n)})$ to the pseudo-group jets.

Moving frames are constructed through a normalization procedure based on a choice of a *cross-section* $K^{(n)}$ to the pseudo-group orbits which we typically choose by fixing the values of r_n of the coordinates $(x, u^{(n)})$. Then the group component $g^{(n)} = \gamma^{(n)}(x, u^{(n)})$ of the moving frame is determined by the condition that $g^{(n)} \cdot (x, u^{(n)}) \in K^{(n)}$.

With a moving frame at hand, the *invariantization* of a function, differential form, etc., is obtained by replacing the pseudo-group parameters by their moving frame expressions in the transform of the object under the pseudo-group action. In particular, invariantizing the coordinate functions on J^n leads to the *normalized differential invariants*

$$H^i = \iota(x^i), \quad i = 1, \ldots, p, \qquad I^\alpha_J = \iota(u^\alpha_J), \quad \alpha = 1, \ldots, q, \quad \#J \geq 0, \quad (6)$$

collectively denoted by $(H, I^{(n)}) = \iota(x, u^{(n)})$. Of these, those corresponding to the constant coordinates determining $K^{(n)}$ will be the constant *phantom differential invariants*, while the remaining *basic differential invariants* form a complete system of functionally independent differential invariants of order $\leq n$ for the prolonged pseudo-group action on submanifolds.

Secondly, invariantization of the basis horizontal and contact one-forms

$$\varpi^i = \iota(dx^i), \qquad \vartheta^\alpha_K = \iota(\theta^\alpha_K), \qquad (7)$$

$i = 1, \ldots, p$, $\alpha = 1, \ldots, q$, $\#K \geq 0$, under a complete moving frame ρ^∞ leads to an invariant coframe on J^∞. The associated total differential operators $\mathcal{D}_1, \ldots, \mathcal{D}_p$ dual to ϖ^i commute with the pseudo-group action and, consequently, map differential invariants to new differential invariants.

5. Recurrence Formulae

The recurrence formulae connect differentiated invariants and forms with their normalized counterparts. Remarkably, they are established, through just linear algebra and differentiation, using only the formulas for the infinitesimal determining equation and the cross-section.

Let $\nu^{(\infty)} = (\rho^{(\infty)})^* \mu^{(\infty)}$ denote the pulled-back Maurer–Cartan forms, which, in view of Theorem 3.1, are subject to the linear relations

$$L^{(n)}(H, I, \nu^{(n)}) = \iota\big[\, L^{(n)}(z, \zeta^{(n)}) \,\big] = 0, \qquad n \geq 0, \qquad (8)$$

where we set $\iota(\zeta^b_A) = \nu^b_A$. Let

$$\eta^i = \iota(\xi^i) = \nu^i, \qquad \widehat{\psi}^\alpha_J = \iota(\widehat{\varphi}^{\,\alpha}_J) = \Phi^\alpha_J(I^{(n)}, \nu^{(n)}), \qquad (9)$$

denote the invariantizations of the standard components of the prolongation $\mathbf{v}^{(\infty)}$ of the vector field

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x, u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \varphi^\alpha(x, u) \frac{\partial}{\partial u^\alpha} \in \mathcal{X}(M).$$

With these in hand, the desired *universal recurrence formula* is as follows.

Theorem 5.1. *If Ω is any differential form on J^∞, then*

$$d\,\iota(\Omega) = \iota\big[\, d\,\Omega + \mathbf{v}^{(\infty)}(\Omega)\,\big], \tag{10}$$

where $\mathbf{v}^{(\infty)}(\Omega)$ denotes the Lie derivative of Ω with respect to the prolonged vector field $\mathbf{v}^{(\infty)}$.

When applied to the normalized invariants, equation (10) directly yields the formulas

$$\begin{aligned} dH^i &= \iota\big(\, dx^i + \xi^i\,\big) = \varpi^i + \eta^i, \\ dI^\alpha_J &= \iota\big(\, du^\alpha_J + \widehat{\varphi}^{\,\alpha}_{\,J}\,\big) = \sum_{i=1}^{p} I^\alpha_{J,i}\,\varpi^i + \vartheta^\alpha_J + \widehat{\psi}^\alpha_J. \end{aligned} \tag{11}$$

Theorem 5.2. *Suppose the pseudo-group admits a moving frame on $\mathcal{V}^n \subset \mathrm{J}^n$, then the n^{th} order recurrence formulae corresponding to the phantom invariants can be uniquely solved to express the pulled-back Maurer–Cartan forms ν^b_A of order $\#A \leq n$ as invariant linear combinations of the invariant horizontal and contact one-forms $\varpi^i, \vartheta^\alpha_J$.*

Substituting the resulting expressions into the non-phantom formulae in (11) leads to a complete system of recurrence relations for the basic differential invariants. In particular, the horizontal components of these produce the equations

$$\mathcal{D}_i H^j = \delta^j_i + M^j_i, \qquad \mathcal{D}_i I^\alpha_J = I^\alpha_{J,i} + M^\alpha_{J,i}, \tag{12}$$

where δ^j_i is the Kronecker delta and M^j_i, $M^\alpha_{J,i}$ are the *correction terms*. One complication, to be dealt with in section 6, is that the correction terms $M^\alpha_{J,i}$ can have the same order as the initial differential invariant $I^\alpha_{J,i}$.

6. The Symbol Modules

To avoid technical complications, we will work in the analytic category in this section. Let (4) be the formally integrable completion of the linearized determining equations of a pseudo-group $\mathcal{G}$. At each $z \in M$, we let $\mathcal{I}|_z$ denote the *symbol module* of the determining equations, which, by formal integrability, forms a submodule of $\mathcal{T} \simeq \mathbb{R}[t] \otimes \mathbb{R}^m$ consisting of real polynomials $\eta(t,T) = \sum_{a=1}^m \eta_a(t)\, T^a$ in $t = (t_1, \ldots, t_m)$ and $T = (T^1, \ldots, T^m)$ that are linear in the T's.

Analogously, let $\widehat{\mathcal{S}} \simeq \mathbb{R}[s] \otimes \mathbb{R}^q$ denote the module consisting of polynomials $\widehat{\sigma}(s,S) = \sum_{\alpha=1}^q \widehat{\sigma}_\alpha(s)\, S^\alpha$ in $s = (s_1, \ldots, s_p)$, $S = (S^1, \ldots, S^q)$, which are linear in the S's. At each submanifold 1-jet $z^{(1)} = (x, u^{(1)}) \in \mathrm{J}^1(M,p)$, we define a linear map $\boldsymbol{\beta}|_{z^{(1)}} : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ by the formulas

$$\begin{aligned} s_i &= \beta_i(z^{(1)};t) = t_i + \sum_{\alpha=1}^{q} u_i^\alpha\, t_{p+\alpha}, && i = 1,\ldots,p, \\ S^\alpha &= B^\alpha(z^{(1)};T) = T^{p+\alpha} - \sum_{i=1}^{p} u_i^\alpha\, T^i, && \alpha = 1,\ldots,q. \end{aligned} \tag{13}$$

Definition 6.1. The *prolonged symbol submodule* at $z^{(1)} \in \mathrm{J}^1|_z$ is the inverse image of the symbol module under the pull-back map $(\boldsymbol{\beta}|_{z^{(1)}})^*$:

$$\mathcal{J}|_{z^{(1)}} = ((\boldsymbol{\beta}|_{z^{(1)}})^*)^{-1}(\mathcal{I}|_z) = \left\{ \sigma(s,S) \;\middle|\; (\boldsymbol{\beta}|_{z^{(1)}})^*(\sigma) \in \mathcal{I}|_z \right\} \subset \widehat{\mathcal{S}}\,. \tag{14}$$

It can be proved that, when the pseudo-group admits a moving frame, the module $\mathcal{J}|_{z^{(1)}}$ coincides with the symbol module associated with the prolonged infinitesimal generators.

To relate this construction to the differential invariant algebra, we invariantize the modules using a moving frame. In general, the invariantization of a prolonged symbol polynomial

$$\sigma(x,u^{(1)};s,S) = \sum_{\alpha=1}^{q} \sum_{\#J \geq 0} h_\alpha^J(x,u^{(1)})\; s_J S^\alpha \in \mathcal{J}|_{z^{(1)}},$$

is given by

$$\widetilde{\sigma}(H,I^{(1)};s,S) = \iota\big[\,\sigma(x,u^{(1)};s,S)\,\big] = \sum_{\alpha=1}^{q} \sum_{\#J \geq 0} h_\alpha^J(H,I^{(1)})\; s_J S^\alpha, \tag{15}$$

which we identify with the differential invariant

$$I_{\tilde\sigma} = \sum_{\alpha=1}^{q} \sum_{\#J \geq 0} h_\alpha^J(H,I^{(1)})\, I_J^\alpha.$$

Let $\widetilde{\mathcal{J}}\,|_{(H,I^{(1)})} = \iota(\mathcal{J}|_{z^{(1)}})$ denote the *invariantized prolonged symbol submodule.*

The recurrence formulae for the differential invariants $I_{\tilde\sigma}$ take the form

$$\mathcal{D}_i\, I_{\tilde\sigma} = I_{s_i\,\tilde\sigma} + M_{\tilde\sigma,i}, \tag{16}$$

in which, unlike in (12), when $\deg\widetilde{\sigma} \gg 0$, the *leading term* $I_{s_i\,\tilde\sigma}$ is strictly of higher order that the *correction term.* Now iteration of 16 leads to the Constructive Basis Theorem for differential invariants.

Theorem 6.1. *Let $\mathcal{G}$ be a Lie pseudo-group admitting a moving frame on an open subset of the submanifold jet bundle at order $n^\star$. Then a finite generating system for its algebra of local differential invariants is given by:*

a) the differential invariants $I_\nu = I_{\sigma_\nu}$, where $\sigma_1, \ldots, \sigma_l$ form a Gröbner basis for the invariantized prolonged symbol submodule $\widetilde{\mathcal{J}}$, and, possibly,
b) a finite number of additional differential invariants of order $\leq n^\star$.

We are also able to exhibit a finite generating system of differential invariant syzygies. First, owing to the non-commutative nature of the the invariant differential operators $\mathcal{D}_i$, we have the *commutator syzygies*

$$\mathcal{D}_J I_{\tilde{\sigma}} - \mathcal{D}_{\widetilde{J}} I_{\tilde{\sigma}} = M_{\tilde{\sigma},J} - M_{\tilde{\sigma},\widetilde{J}} \equiv N_{J,\widetilde{J},\tilde{\sigma}}, \quad \text{whenever} \quad \widetilde{J} = \pi(J) \qquad (17)$$

for some permutation π. Provided $\deg \widetilde{\sigma} > n^\star$, the right hand side $N_{J,\widetilde{J},\tilde{\sigma}}$ is of lower order than the terms on the left hand side.

In addition, any algebraic syzygy satisfied by polynomials in $\widetilde{\mathcal{J}}\,|_{(H,I^{(1)})}$ provides an additional syzygy amongst the differentiated invariants. In detail, to each invariantly parametrized polynomial $q(H, I^{(1)}; s) = \sum_J q_J(H, I^{(1)}) s_J \in \mathbb{R}[s]$ we associate an invariant differential operator

$$q(H, I^{(1)}; \mathcal{D}) = \sum_J q_J(H, I^{(1)}) \mathcal{D}_J, \qquad (18)$$

where the sum ranges over non-decreasing multi-indices. In view of (16), whenever $\widetilde{\sigma}(H, I^{(1)}; s, S) \in \widetilde{\mathcal{J}}\,|_{(H,I^{(1)})}$, we can write

$$q(H, I^{(1)}; \mathcal{D})\, I_{\tilde{\sigma}(H,I^{(1)};s,S)} = I_{q(H,I^{(1)};s)\,\tilde{\sigma}(H,I^{(1)};s,S)} + R_{q,\tilde{\sigma}}, \qquad (19)$$

where $R_{q,\tilde{\sigma}}$ has order $<$ $\deg q + \deg \widetilde{\sigma}$. Thus, any algebraic syzygy $\sum_{\nu=1}^{l} q_\nu(H, I^{(1)}, s)\, \sigma_\nu(H, I^{(1)}; s, S) = 0$ of the Gröbner basis polynomials of $\widetilde{\mathcal{J}}\,|_{(H,I^{(1)})}$ induces a syzygy among the generating differential invariants, $\sum_{\nu=1}^{l} q_\nu(H, I^{(1)}, \mathcal{D})\, I_{\tilde{\sigma}_\nu} = R$, where order$R < \max\{\deg q_\nu + \deg \widetilde{\sigma}_\nu\}$.

Theorem 6.2. *Every differential syzygy among the generating differential invariants is a combination of the following:*

(a) the syzygies among the differential invariants of order $\leq n^\star$,
(b) the commutator syzygies,
(c) syzygies coming from an algebraic syzygy among the Gröbner basis polynomials.

In this manner, we deduce a finite system of generating differential syzygies for the differential invariant algebra of our pseudo-group.

Further details and applications of these results can be found in our papers listed in the references.

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