Oregon State University
Mathematics Department

Study Guide and Lab Manual
for
Mth 252 – Integral Calculus
to accompany the text

*Calculus*

*Early Transcendentals*

Second Edition
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Lyle Cochran
Bernard Gillett

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(with significant contributions from D. Hart
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†Optional
Introduction and Notes to Students

Introduction:

This study guide is set up for a ten week term with 29 lectures and 9 to 10 recitations. Please check with your instructor for a detailed syllabus for your section of the course.

Prerequisites:

It is expected that you are comfortable with differential calculus. This includes familiarity with material from previous courses, such as basic algebraic manipulations, including those involving fractions and exponents, as well as trigonometry. The most common difficulty for students in this course is with “old” concepts such as algebra; if your skills are rusty, practice now, and get help early.

Comments and suggestions on study habits:

There is much new material presented in this course. It is very important to develop good study habits and to keep up with the course material right from the beginning of the course. Listed below are a number of things to watch out for.

Time: It is very important to devote enough time on a daily basis for the course. You should plan on spending two to three hours going over the material and assignments for each hour of lecture. If you have seen some of this material before, it is easy to fall into the pattern of not devoting enough time to the material at the beginning of the course. When more difficult material is presented in a few weeks you then will find yourself too far behind.

Algebra skills: The most common difficulty for students in this course is remembering the correct algebraic manipulations from pre-Calculus courses. If you find yourself having difficulty, get help early in the course. Don't wait until it is too late. The material in the course is based on the assumption that you remember the algebra and trigonometry that you learned in your previous courses.

Attendance: Many students make the mistake of thinking that they can learn all the required material by just working the homework problems without attending the lectures. Often, your lecturer will present an additional viewpoint on the material that is not presented in the text. The exams for the course are based on the text material and on the presentations in lecture. Make sure that you attend all the lectures. If for some reason you miss a lecture, get notes for the missed class from one of the other students in the course.

Homework: Often, students think that they have mastered the material if they can
get the answers in the back of the book. You should look at the answers in the back of the book only after you have completed the assignment. Relying on the solutions can give you a false sense of security.

Your homework assignments may include problems that are not available through the online homework. Do not make the mistake of not doing these problems. They often cover important concepts that are difficult practice in an online environment.

Other Resources:

The Mathematics Learning Center (MLC) provides drop-in help for all lower division mathematics courses. The MLC is located on the ground floor of Kidder Hall in room 108, and is normally open Monday through Thursday from 9 a.m. to 6 p.m. and on Fridays from 9 a.m. to 5 p.m., from the second week of the term through the Dead Week. The MLC also provides evening tutoring in the Valley Library, in general Sunday through Thursday from 6 p.m. till 9 p.m. Current hours can be found at the MLC homepage, http://www.math.oregonstate.edu/mlc

By purchasing the course text, you will gain access to MyMathLab, an online calculus portal maintained by the publisher. In addition to homework problems and tests with automated grading, the site offers a number of useful tools, including Powerpoint and video lectures, review cards, and tutorial exercises, for organizing your studies and to facilitate learning the course material.

http://www.coursecompass.com/

On page 11 there is a table of basic integrals. You should print this table and keep it available during the first couple of weeks of lecture. If you find that you are still referring to this table after this time then you need to spend more time with the flash cards.
Lesson 1
Antiderivatives
Section 4.9

**Definition:** We call $F$ an antiderivative of $f$ on an interval $I$ if $F'(x) = f(x)$ for all $x$ in $I$.

Since $\frac{d}{dx} \left( \frac{x^3}{3} \right) = x^2$, an antiderivative of $x^2$ is $\frac{x^3}{3}$. It is worth noting that $\frac{x^3}{3} + 8$ is also an antiderivative of $x^2$. In fact, $\frac{x^3}{3} + C$ plus any constant is an antiderivative of $x^2$.

From the Mean Value Theorem, the following result was established in differential calculus:

**Uniqueness of Antiderivatives:** If $F'(x) = G'(x)$ for all $x$ in an open interval $(a,b)$, then there is a constant $C$ such that $G(x) = F(x) + C$ for all $x$ in $(a,b)$.

The set of all possible antiderivatives of a function $f$ is denoted as $\int f(x) \, dx$. From the Uniqueness of Antiderivative Theorem,

$$\int x^2 \, dx = \left\{ \frac{x^3}{3} + C \right\} \quad \text{if} \quad C \text{ is any real number}.$$  

This is usually written as

$$\int x^2 \, dx = \frac{x^3}{3} + C$$

where $C$ is interpreted as an arbitrary constant.

$C$ is called the constant of integration. By tradition, $C$ is almost always used as the variable for the constant of integration.

$\int f(x) \, dx$ is called an indefinite integral.

$f$ is called the integrand.

d$x$ is a differential.

$x$ is the dummy variable of integration. Since $x$ is a dummy variable, any otherwise unused variable can be used. The following formulas are equivalent.

$$\int x^2 \, dx = \frac{x^3}{3} + C, \quad \int r^2 \, dt = \frac{r^3}{3} + C, \quad \int \xi^2 \, d\xi = \frac{\xi^3}{3} + C.$$
\[ \int \] is an integral sign. You can think of \[ \int \] as an elongated S for sum. The reason for this will become clear in later sections.

A large portion of this term will be spent on developing and learning methods of finding antiderivatives which we will refer to as techniques of integration.

The first two techniques of integration are using stock formulas and using algebraic manipulation to reduce to stock formulas.

Every differentiation formula will give a corresponding integration formula, some more useful than others. For example, from the differentiation formula

\[
\frac{d}{dx}(e^{kx}) = ke^{kx} \quad \text{where } k \text{ is a constant}
\]

the following integration formula is established:

\[
\int e^{kx} \, dx = \frac{1}{k} e^{kx} + C.
\]

Any integration formula can be verified by differentiation. Since

\[
\frac{d}{dx}\left(\ln|\sec(x) + \tan(x)|\right) = \frac{d}{dx}\left(\frac{\sec(x) + \tan(x)}{\sec(x) + \tan(x)}\right)
= \frac{\sec(x) \tan(x) + \sec^2(x)}{\sec(x) + \tan(x)}
= \sec(x)\left(\frac{\sec(x) + \tan(x)}{\sec(x) + \tan(x)}\right)
= \sec(x)
\]

the integration formula

\[
\int \sec(x) \, dx = \ln|\sec(x) + \tan(x)| + C
\]

is verified.

Here is a table of stock integration formulas. You should verify each of these formulas by differentiation. You should also start learning these formulas.
Basic Integral Table

1. \( \int k f(x) \, dx = k \int f(x) \, dx \)
2. \( \int (f(x) + g(x)) \, dx = \int f(x) \, dx + \int g(x) \, dx \)
3. \( \int (f(x) - g(x)) \, dx = \int f(x) \, dx - \int g(x) \, dx \)
4. \( \int x^p \, dx = \frac{x^{p+1}}{p+1} + C, \quad p \neq -1 \)
5. \( \int \frac{1}{x} \, dx = \ln|x| + C \)
6. \( \int e^{kx} \, dx = \frac{1}{k} e^{kx} + C \)
7. \( \int \ln(x) \, dx = x \ln(x) - x + C \)
8. \( \int \sin(kx) \, dx = -\frac{1}{k} \cos(kx) + C \)
9. \( \int \cos(kx) \, dx = \frac{1}{k} \sin(kx) + C \)
10. \( \int \tan(kx) \, dx = \frac{1}{k} \ln|\sec(kx)| + C \)
11. \( \int \cot(kx) \, dx = -\frac{1}{k} \ln|\csc(kx)| + C \)
12. \( \int \sec(kx) \, dx = \frac{1}{k} \ln|\sec(kx) + \tan(kx)| + C \)
13. \( \int \csc(kx) \, dx = -\frac{1}{k} \ln|\csc(kx) + \cot(kx)| + C \)
14. \( \int \sec^2(kx) \, dx = \frac{1}{k} \tan(kx) + C \)
15. \( \int \csc^2(kx) \, dx = -\frac{1}{k} \cot(kx) + C \)
16. \( \int \sec(kx) \tan(kx) \, dx = \frac{1}{k} \sec(kx) + C \)
17. \( \int \csc(kx) \cot(kx) \, dx = -\frac{1}{k} \csc(kx) + C \)
18. \( \int \frac{dx}{x^2 + k^2} = \frac{1}{k} \tan^{-1} \left( \frac{x}{k} \right) + C, \quad k > 0 \)
19. \( \int \frac{dx}{\sqrt{k^2 - x^2}} = \sin^{-1} \left( \frac{x}{k} \right) + C, \quad k > 0, \ |x| < k \)
20. \( \int \frac{dx}{x \sqrt{x^2 - k^2}} = \frac{1}{k} \sec^{-1} \left| \frac{x}{k} \right| + C, \quad k > 0, \ |x| > k \)
Often algebraic manipulation can be used to manipulate your integral into a form that can be evaluated by stock formulas.

Example 1:
\[
\int \frac{x^5 + 5}{x^2} \, dx = \int \left( x^3 + \frac{5}{x^2} \right) \, dx = \int x^3 \, dx + 5 \int x^{-2} \, dx = \frac{x^4}{4} + \frac{5}{2x} + C
\]
\[
= \frac{x^4}{4} - \frac{5}{x} + C
\]

Example 2:
\[
\int \frac{dx}{4x^2 + 25} = \frac{1}{4} \int \frac{dx}{x^2 + \left(\frac{5}{2}\right)^2} = \frac{1}{4} \cdot \frac{1}{\frac{5}{2}} \tan^{-1}\left(\frac{x}{\frac{5}{2}}\right) + C
\]
\[
= \frac{1}{10} \tan^{-1}\left(\frac{2x}{5}\right) + C
\]

Example 3:
\[
\int \left( x^2 + 3 \right)^2 \, dx = \int \left( x^4 + 6x^2 + 9 \right) \, dx = \int x^4 \, dx + 6 \int x^2 \, dx + 9 \int dx
\]
\[
= \frac{x^5}{5} + 3x^3 + 9x + C
\]

NOTE: \( \int 1 \, dx \) is usually written as \( \int dx \).

The following 9 pages can be printed and used as flash cards. Cut along the solid lines and fold on the dotted line. You can also cut on the dotted line and tape to 3 by 5 index cards.

**Suggested Homework:**
Note: The assignment given by your instructor may be different.

Read Pages 318 – 329
Problems Available Online: Pages 327 – 329 – \{ 3, 11, 17, 25, 37, 41, 49, 52, 69, 85, 101, 107, 119 \}
Problems from textbook: Pages 327 – 329 – \{ 2, 98, 114 \}
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<td>1. $= \frac{x^{p+1}}{p+1} + C$</td>
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<td>2. $\int \frac{1}{x} , dx$</td>
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<td>3. $\int e^{kx} , dx$</td>
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<td>4. $\int \ln(x) , dx$</td>
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<td>5. $\int \sin(kx) , dx$</td>
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| 7. | \[
\int \tan(kx) \, dx = \frac{1}{k} \ln|\sec(kx)| + C
\] |
| 8. | \[
\int \cot(kx) \, dx = -\frac{1}{k} \ln|\csc(kx)| + C
\] |
| 9. | \[
\int \sec(kx) \, dx = \frac{1}{k} \ln|\sec(kx) + \tan(kx)| + C
\] |
| 10. | \[
\int \csc(kx) \, dx = -\frac{1}{k} \ln|\csc(kx) + \cot(kx)| + C
\] |
| 11. | \[
\int \sec^2(kx) \, dx = \frac{1}{k} \tan(kx) + C
\] |
| 12. | \[
\int \csc^2(kx) \, dx = -\frac{1}{k} \cot(kx) + C
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   |   | \[ = \frac{1}{k} \sec(kx) + C \]  
| 14. | \[ \int \csc(kx) \cot(kx) \, dx \]  
   |   | \[ = -\frac{1}{k} \csc(kx) + C \]  
| 15. | \[ \int \frac{dx}{x^2 + k^2}, \quad k > 0 \]  
   |   | \[ = \frac{1}{k} \tan^{-1} \left( \frac{x}{k} \right) + C \]  
| 16. | \[ \int \frac{dx}{\sqrt{k^2 - x^2}}, \quad k > 0 \]  
   |   | \[ = \sin^{-1} \left( \frac{x}{k} \right) + C \]  
| 17. | \[ \int \frac{dx}{x \sqrt{x^2 - k^2}}, \quad k > 0 \]  
   |   | \[ = \frac{1}{k} \sec^{-1} \left| \frac{x}{k} \right| + C \]  
| 18. | \[ \int e^x \, dx \]  
   |   | \[ = e^x + C \]  

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<td>( \int \cot(x) , dx )</td>
<td>22.</td>
<td>( = -\ln</td>
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<td>23.</td>
<td>( \int \sec(x) , dx )</td>
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<td>( = \ln</td>
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<tr>
<td>24.</td>
<td>( \int \csc(x) , dx )</td>
<td>24.</td>
<td>( = -\ln</td>
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<td>Equation</td>
<td>Result</td>
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<tr>
<td>$\int \sec^2(x) , dx$</td>
<td>$= \tan(x) + C$</td>
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<tr>
<td>$\int \csc^2(x) , dx$</td>
<td>$= -\cot(x) + C$</td>
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<tr>
<td>$\int \sec(x)\tan(x) , dx$</td>
<td>$= \sec(x) + C$</td>
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<tr>
<td>$\int \csc(x)\cot(x) , dx$</td>
<td>$= -\csc(x) + C$</td>
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<tr>
<td>$\int \frac{dx}{x^2 + 1}$</td>
<td>$= \tan^{-1}(x) + C$</td>
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<tr>
<td>$\int \frac{dx}{\sqrt{1-x^2}}$</td>
<td>$= \sin^{-1}(x) + C$</td>
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<td></td>
<td>( \int \frac{dx}{x \sqrt{x^2 - 1}} )</td>
<td>31. ( \frac{dx}{x \sqrt{x^2 - 1}} )</td>
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<td>31. ( = \sec^{-1}</td>
<td>x</td>
</tr>
<tr>
<td>32.</td>
<td>( \int (3t + 5) , dt )</td>
<td>32. ( \frac{3}{2}t^2 + 5t + C )</td>
<td></td>
</tr>
<tr>
<td>33.</td>
<td>( \int \frac{dz}{z^3} )</td>
<td>33. ( \int z^{-3} , dz = -\frac{1}{2}z^{-2} + C )</td>
<td></td>
</tr>
<tr>
<td>34.</td>
<td>( \int 5x^{-1} , dx )</td>
<td>34. ( = 5 \ln</td>
<td>x</td>
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<td>35.</td>
<td>( \int \sqrt{x} , dx )</td>
<td>35. ( = \int x^{\frac{1}{2}} , dx = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{2}{3}x^{\frac{3}{2}} + C )</td>
<td></td>
</tr>
<tr>
<td>36.</td>
<td>( \int t^7 , dt )</td>
<td>36. ( = \frac{t^8}{8} + C )</td>
<td></td>
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<tr>
<td>37. ( \int e^{-x} , dx )</td>
<td>37. ( = -e^{-x} + C )</td>
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<tr>
<td>38. ( \int e^{3\theta} , d\theta )</td>
<td>38. ( = \frac{1}{3} e^{3\theta} + C )</td>
<td></td>
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</tr>
<tr>
<td>39. ( \int \sin(5\theta) , d\theta )</td>
<td>39. ( = -\frac{1}{5} \cos(5\theta) + C )</td>
<td></td>
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</tr>
<tr>
<td>40. ( \int \cos(4\beta) , d\beta )</td>
<td>40. ( = \frac{1}{4} \sin(4\beta) + C )</td>
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</tr>
<tr>
<td>41. ( \int \tan(6x) , dx )</td>
<td>41. ( = \frac{1}{6} \ln</td>
<td>\sec(6x)</td>
<td>+ C )</td>
</tr>
<tr>
<td>42. ( \int \cot(\pi x) , dx )</td>
<td>42. ( = -\frac{1}{\pi} \ln</td>
<td>\csc(\pi x)</td>
<td>+ C )</td>
</tr>
<tr>
<td>Problem</td>
<td>Equation</td>
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<tr>
<td>43. $\int \sec(\pi \theta) , d\theta$</td>
<td>$\frac{1}{\pi} \ln</td>
<td>\sec(\pi \theta) + \tan(\pi \theta)</td>
<td>+ C$</td>
</tr>
<tr>
<td>44. $\int \csc(3x) , dx$</td>
<td>$\frac{1}{3} \ln</td>
<td>\csc(3x) + \cot(3x)</td>
<td>+ C$</td>
</tr>
<tr>
<td>45. $\int \sec^2(\pi x) , dx$</td>
<td>$\frac{1}{\pi} \tan(\pi x) + C$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>46. $\int \csc^2(3t) , dt$</td>
<td>$\frac{1}{3} \cot(3t) + C$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>47. $\int \sec(5\alpha) \tan(5\alpha) , d\alpha$</td>
<td>$\frac{1}{5} \sec(5\alpha) + C$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>48. $\int \csc(2x) \cot(2x) , dx$</td>
<td>$\frac{1}{2} \csc(2x) + C$</td>
<td></td>
<td></td>
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<tr>
<td></td>
<td>Equation</td>
<td>Solution</td>
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<tr>
<td>49.</td>
<td>[ \int \frac{dx}{x^2 + 25} ]</td>
<td>[ = \frac{1}{5} \tan^{-1} \left( \frac{x}{5} \right) + C ]</td>
<td></td>
</tr>
<tr>
<td>50.</td>
<td>[ \int \frac{dx}{\sqrt{4 - x^2}} ] \ dx</td>
<td>[ = \sin^{-1} \left( \frac{x}{2} \right) + C ]</td>
<td></td>
</tr>
<tr>
<td>51.</td>
<td>[ \int \frac{dx}{x\sqrt{x^2 - 9}} ] \ dx, \ k &gt; 0</td>
<td>[ = \frac{1}{3} \sec^{-1} \left</td>
<td>\frac{x}{3} \right</td>
</tr>
</tbody>
</table>
Lesson 2

Riemann Sums

Section 5.1

Sigma notation is a shorthand method of representing sums. This notation will be used in the development of Riemann sums.

\[ \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n \]

For example,

\[ \sum_{k=1}^{n} (k^2 + 1) = (1^2 + 1) + (2^2 + 1) + (3^2 + 1) + (4^2 + 1) + (5^2 + 1) \]
\[ = 2 + 5 + 10 + 17 + 26 \]
\[ = 60 \]

Make sure you know how to use sigma notation. It will be used throughout the course.

The formulas below are quite useful.

\[ \sum_{i=1}^{n} 1 = n \]
\[ \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \]
\[ \sum_{i=1}^{n} i^3 = \frac{n(n+1)^2}{4} \]
\[ \sum_{i=1}^{n} i^4 = \frac{n(n+1)(6n^3 + 9n^4 + n - 1)}{30} \]

The second formula is easily verified by the following observation:

\[ \frac{1 + 2 + 3 + \cdots + n}{n + (n-1) + (n-2) + \cdots + 1} = \frac{(n+1)(n+1) + (n+1) + \cdots + (n+1)}{(n+1) + (n+1) + (n+1) + \cdots + (n+1)} = n(n+1) \]
Consider the function $f(x) = \sqrt{x}$ on the interval $0 \leq x \leq 4$. Our interest is in finding the area that is under this curve and above the $x$-axis. Since this region is not one of the regions whose area can be found by elementary geometry, we first estimate this area by approximation by rectangles.

Below are pictures of three such approximations.

The first approximation is a *Left Riemann sum.*
The second approximation is a *Right Riemann sum.*
The third approximation is a *Midpoint Riemann sum.*

Let $f$ be a function that is defined on the closed interval $[a,b]$. To form a Riemann sum, pick a positive integer $n$ and subdivide the interval $[a,b]$ into $n$ subintervals of equal length $\Delta x = \frac{b-a}{n}$. Let $x_0 = a$ and $x_n = b$. There are now $n$ subintervals, $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$, where:

\[
x_0 = a \\
x_1 = a + \Delta x = a + \frac{b-a}{n} \\
x_2 = x_1 + \Delta x = a + 2 \cdot \frac{b-a}{n} \\
\vdots \\
x_i = x_{i-1} + \Delta x = a + i \cdot \frac{b-a}{n} \\
\vdots \\
x_n = x_{n-1} + \Delta x = a + n \cdot \frac{b-a}{n} = b
\]
Pick any point \( \bar{x}_i \) from the interval \([x_{i-1}, x_i]\). Then form a rectangle with height \( f(\bar{x}_i) \) and base \([x_{i-1}, x_i]\). The area of this rectangle is \( f(\bar{x}_i) \Delta x \). Sum the area of these rectangles to form a Riemann sum:
\[
\sum_{i=1}^{n} f(\bar{x}_i) \Delta x.
\]

If \( \bar{x}_i \) is chosen to be the left endpoint of the interval \([x_{i-1}, x_i]\), i.e. \( \bar{x}_i = x_{i-1} \), a Left Riemann sum is formed.
\[
L_n = \sum_{i=1}^{n} f(x_{i-1}) \Delta x
\]

If \( \bar{x}_i \) is chosen to be the right endpoint of this interval, a Right Riemann sum is found.
\[
R_n = \sum_{i=1}^{n} f(x_i) \Delta x
\]

This Midpoint Riemann sum is found when \( \bar{x}_i \) is taken to be the midpoint of the interval.
\[
M_n = \sum_{i=1}^{n} f\left(\frac{x_i + x_{i-1}}{2}\right) \Delta x
\]

To improve the accuracy of the approximation, take a larger value of \( n \). Since a common definition of Calculus is the study of limits, it is clear what will happen to \( n \) in the next section.

**Suggested Homework:**
Note: The assignment given by your instructor may be different.

Read Pages 333 – 347
Problems Available Online: Pages 343 – 347 – \{ 1, 5, 7, 9, 24, 28, 41, 65, 71 \}
Problems from textbook: Pages 343 – 347 – \{ 34(i) – a through h, 43(a) – find the sigma notation for the right, left and midpoint Riemann sums, no need to evaluate with a calculator \}
Lesson 3
Definite Integrals
Section 5.2

When a real world situation can be modeled by an equation involving Riemann sums, the solution often involves evaluating integrals. This will be a common theme throughout this course.

**Definition:** A general Riemann sum of a function $f$ defined on an interval $[a,b]$ is defined as

$$
\sum_{k=1}^{n} f(\bar{x}_k) \Delta x_k
$$

where $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$ is a partition of $[a,b]$, $x_{k-1} \leq \bar{x}_k \leq x_k$, and $\Delta x_k = x_k - x_{k-1}$.

Let $\Delta = \max \{\Delta x_1, \Delta x_2, \Delta x_3, \cdots, \Delta x_n\}$.

**Definition:** The definite integral of a function $f$ defined on an interval $[a,b]$ is defined as

$$
\int_{a}^{b} f(x) \ dx = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(\bar{x}_k) \Delta x_k
$$

provided this limit exists for all choices of partitions of $[a,b]$ and all choices of $\bar{x}_k$ on a partition.

The key idea is that a definite integral is a limit of Riemann sums. The fact that this limit exists for all functions that are continuous on the interval $[a,b]$ is usually proven in an advanced Calculus course. The main idea of the proof depends heavily on the Extreme Value Theorem.

While the process of evaluating a definite integral by evaluating the limit of a Riemann sum is generally rather difficult and is well beyond the scope of this course, there are several simple examples that can be done. Make sure that you carefully read and understand the process. You are given the opportunity in the homework to work out a few examples. You will need the summation formulas given in the last section to do these problems. You may also want to review limits to infinity.
Many definite integrals can be evaluated by elementary geometry. Explain why
\[ \int_{-1}^{1} \sqrt{1-x^2} \, dx = \frac{\pi}{2}. \]

You may have already taken notice that the integral sign \( \int \) is used for both indefinite and definite integrals. The relationship between antiderivatives and definite integrals will be made clear in the next section.

**Suggested Homework:**
Note: The assignment given by your instructor may be different.

Read Pages 348 – 361
Problems Available Online: Pages 358 – 361 – \{ 5, 7, 21, 26, 29, 31, 35, 43, 45, 50, 51, 78 \}
Problems from textbook: Pages 358 – 361 – \{ 67, 72, 77 \}
Lesson 4

Fundamental Theorem of Calculus

Section 5.3

In Lesson 4, the definite integral, \( \int_a^b f(x) \, dx \), was defined as the limit of Riemann sums.

In Lesson 2, the indefinite integral, \( \int f(x) \, dx \), was defined to be the set of all antiderivatives of \( f \). In this lesson, the use of the integral sign and the term integral for both of these concepts will be justified.

Given a function \( f \) defined on an interval that contains the point \( a \), define the area function as

\[
A(x) = \int_a^x f(t) \, dt
\]

Recall that \( t \) in this expression is the dummy variable of integration. The variable \( x \) cannot be used since \( x \) is already being used as the variable for the area function.

Consider the following calculation:

\[
\frac{d}{dx} A(x) = \lim_{h \to 0} \frac{A(x + h) - A(x)}{h} = \lim_{h \to 0} \frac{\int_a^{x+h} f(t) \, dt - \int_a^x f(t) \, dt}{h}
\]

\[
= \lim_{h \to 0} \int_x^{x+h} \frac{f(t) \, dt}{h}
\]

Justify this step.

If \( f \) is continuous at \( x \) and \( h \) is small, then \( \int_x^{x+h} f(t) \, dt \) is approximately \( f(x)h \).

From this it is seen that it is quite reasonable to seek a formal proof that \( \frac{d}{dx} A(x) = f(x) \).

This is the substance of the next result.

**Fundamental Theorem of Calculus – Part 1**

If \( f \) is function that is continuous on an interval \( I \) that contains the point \( a \), then the function \( A(x) = \int_a^x f(t) \, dt \) is differentiable on \( I \) and

\[
\frac{d}{dx} A(x) = \frac{d}{dx} \int_a^x f(t) \, dt = f(x)
\]
Notice the name of this theorem, **Fundamental Theorem of Calculus**. This should give you some idea of the importance of this result.

The Fundamental Theorem of Calculus – Part 1 shows that the function

\[ A(x) = \int_a^x f(t) \, dt \] is an antiderivative of \( f \). Let \( F \) be any other antiderivative of \( f \).

Then there is a constant \( C \) where \( \int_a^x f(t) \, dt = F(x) + C \). In the above equation, let \( x = a \) and make the observation that \( \int_a^a f(t) \, dt = 0 \) to see that

\[
\int_a^a f(t) \, dt = F(a) + C \\
0 = F(a) + C \\
\text{Then } C = -F(a).
\]

Then \( \int_a^x f(t) \, dt = F(x) - F(a) \). Take \( x = b \) to obtain the next result.

**Fundamental Theorem of Calculus – Part 2**

If \( f \) is continuous on the interval \([a,b]\) and \( F \) is any antiderivative of \( f \) on \([a,b]\), then

\[
\int_a^b f(x) \, dx = F(b) - F(a)
\]

Notice that this result, which follows from Part 1, is also called the Fundamental Theorem of Calculus (FTC).

There is a shorthand notation that is commonly used.

\[
\int_a^b f(x) \, dx = F(b) - F(a)
\]

The use of the integral sign for both definite and indefinite integrals is now justified.

**Suggested Homework:**

Note: The assignment given by your instructor may be different.

Read Pages 362 – 376

Problems Available Online: Pages 373 – 376 – \{ 11, 18, 25, 29, 44, 57, 61, 63, 69, 75, 89, 93 \}

Problems from textbook: Pages 373 – 376 – \{ 46, 85 \}
Lesson 5
Working with Integrals
Section 5.4

Recall that a function is even if for all $x$ in the domain of $f$, $f(-x) = f(x)$. The graph of an even function is symmetric about the y-axis. Examples of even functions are $f(x) = \cos(x)$ and $f(x) = 3x^4 + 2x^2 - 5$. If an even function $f$ is defined on the interval $[-a,a]$, then $\int_{-a}^{a} f(x) \, dx = 2\int_{0}^{a} f(x) \, dx$.

An odd function satisfies the property that for all $x$ in the domain of $f$, $f(-x) = -f(x)$. The graph of an odd function is symmetric about the origin. Examples of odd functions are $f(x) = \sin(x)$ and $f(x) = x^3 - 4x$. If an odd function $f$ is defined on the interval $[-a,a]$, then $\int_{-a}^{a} f(x) \, dx = 0$.

By noticing that a function has symmetry, calculations can often be simplified.

The average value of a function defined on the interval $[a,b]$ is defined as,

$$ \bar{f} = \frac{1}{b-a} \int_{a}^{b} f(x) \, dx. $$

If you think of the area between the graph of $f$ and the $x$-axis as a thin film of water magically being held in place, then the average value of a function will be the height of the water when the magic is removed.

If $f$ is continuous on the interval $[a,b]$, then there will be a point $c$ with $a \leq c \leq b$ where $f(c)$ is equal to the average value of $f$ on $[a,b]$. This result is called the Mean Value Theorem for Integrals.

Suggested Homework:
Note: The assignment given by your instructor may be different.

Read Pages 377 – 384
Problems Available Online: Pages 381 – 384 – \{ 3, 9, 25, 33, 35, 41 \}
Problems from textbook: Pages 381 – 384 – \{ 15 \}
Lesson 6
Substitution
Section 5.5

Recall that every differentiation formula has a corresponding integration formula. The formula that corresponds to the chain rule,

$$\frac{d}{dx} F(g(x)) = F'(g(x))g'(x) = f(g(x))g'(x) ,$$

where $F$ is an antiderivative of $f$, we obtain the integration formula,

$$\int f(g(x))g'(x) \ dx = F(g(x)) + C .$$

This leads to a technique of integration that is called either substitution or change of variables. This technique is often called $u$ substitution since $u$ is commonly used as the substitution variable.

To use this method you need to select a potential substitution $u = g(x)$ and then calculate $du = g'(x) \ dx$. Then

$$\int f(g(x))g'(x) \ dx = \int f(u) \ du = F(u) + C = F(g(x)) + C .$$

Finding the proper substitution comes from experience and from trial and error. As you gain more experience through practice, you will have to resort to trial and error less often. It is important to note that not all integrals can be evaluated through substitution. Several techniques of integration will be covered this term.

It is unfortunately true that not every simple function has an antiderivative that can be expressed in terms of elementary functions. An example of such an integral is

$$\int e^{-x^2} \ dx .$$

When substitution is used with definite integrals, the limits of integration also change.

$$\int_a^b f(g(x))g'(x) \ dx = \int_{g(a)}^{g(b)} f(u) \ du = F(g(b)) - F(g(a))$$
You can avoid changing the limits of integration by converting back to the original variable.

\[
\int_{a}^{b} f(g(x))g'(x) \, dx = \int_{x=a}^{x=b} f(u) \, du = F(u) \bigg|_{x=a}^{x=b} = F(g(b)) - F(g(a))
\]

There are several trigonometric identities that are useful in evaluating integrals.

\[
\begin{align*}
\sin^2 x &= \frac{1}{2}(1 - \cos(2x)) \\
\cos^2 x &= \frac{1}{2}(1 + \cos(2x)) \\
\sin x \cos x &= \frac{1}{2}\sin(2x)
\end{align*}
\]

Here’s an example.

\[
\int \sin^2 x \cos^2 x \, dx = \int \left(\sin x \cos x\right)^2 \, dx
\]

\[
= \int \left(\frac{1}{2}\sin(2x)\right)^2 \, dx \quad \text{substitute } \sin x \cos x = \frac{1}{2}\sin(2x)
\]

\[
= \frac{1}{4} \int \sin^2(2x) \, dx
\]

\[
= \frac{1}{8} \int 1 - \cos(4x) \, dx \quad \text{substitute } \sin^2(2x) = \frac{1}{2}(1 - \cos(4x))
\]

\[
= \frac{1}{8} x - \frac{1}{32} \sin(4x) + C
\]

**Suggested Homework:**
Note: The assignment given by your instructor may be different.

Read Pages 384 – 393

Indefinite Integrals
Problems Available Online: Pages 390 – 393 – { 1, 2, 16, 17, 18, 19, 20, 22, 23, 27, 33, 36 }
Problems from textbook: Pages 390 - 393 – { 4, 7, 8 }

Definite Integrals
Problems Available Online: Pages 390 – 393 – { 6, 39, 41, 43, 45, 57, 69, 79, 81, 99 }
Problems from textbook: Pages 390 – 393 – { 61, 67 }
Lesson 7

Velocity and Net Change

Section 6.1

Let the position of an object moving along the $x$-axis at time $t$ be given by $s(t)$. Then the velocity of that object at time $t$ is $v(t) = s'(t)$ and its acceleration is $a(t) = v'(t) = s''(t)$. Velocity includes both magnitude (speed) and direction of travel. Accordingly, the speed of the object at time $t$ is given by $|v(t)|$.

The displacement of the particle from time $t = a$ to time $t = b$ where $b > a$ is

$$s(b) - s(a) = \int_a^b v(t) \, dt .$$

The total distance travelled by the particle from time $t = a$ to time $t = b$ where $b > a$ is

$$\int_a^b |v(t)| \, dt .$$

Under what conditions will the displacement and the total distance travelled be different?

How can these ideas be applied to other situations?

**Suggested Homework:**
Note: The assignment given by your instructor may be different.

Read Pages 398 – 411

Problems Available Online: Pages 407 – 411 – \{ 10, 19, 25, 39, 41, 43, 51, 61 \}

Problems from textbook: Pages 407 – 411 – \{ 44 \}
Lesson 8

Regions Between Curves

Section 6.2

The area of the region bounded by the curves \( y = f(x) \), \( y = g(x) \), \( x = a \), and \( x = b \) is given by

\[
\int_a^b |f(x) - g(x)| \, dx.
\]

Whenever possible, you should plot the curves to help with the evaluation of the integral. To evaluate this integral, first split the integral up at the locations where \( f(x) = g(x) \). Then over each of these intervals, rewrite the integral by the rule,

\[
|f(x) - g(x)| = \begin{cases} 
  f(x) - g(x), & f(x) \geq g(x) \\
  g(x) - f(x), & f(x) \leq g(x)
\end{cases}
\]

To find the area enclosed between two curves \( y = f(x) \) and \( y = g(x) \), the first step, generally, will be to plot the curves to get a good visualization of the region. To determine the limits of integration, you need to solve the equation \( f(x) = g(x) \). Next you need to determine whether \( f(x) \geq g(x) \) or \( g(x) \geq f(x) \) on this interval. Use this information to set up the proper integral.

On occasion it is useful to interchange the roles of \( x \) and \( y \) and deal with curves of the form \( x = h(y) \) and \( x = k(y) \).

The justification for the use of integrals to find these areas depends of the proper evaluation of a limit of Riemann sums. Make sure you understand this process. Using a limit of Riemann sums to find the integral for a particular application will be a common theme over the next several lessons.

Suggested Homework:
Note: The assignment given by your instructor may be different.

Read Pages 412 – 420

Problems Available Online: Pages 416 – 420 – \{ 5, 11, 16, 29, 35, 69 \}
Problems from textbook: Pages 416 – 420 – \{ 6, 25, 39 \}
Lesson 9

Volume from Cross Sectional Area

Section 6.3

Suppose that you are given a three-dimensional solid that is oriented so that it extends along the $x$-axis from $x = a$ to $x = b$. Also suppose that at each point $x$ in this interval you have the cross sectional area, $A(x)$ of this solid. After carefully setting up the proper Riemann sum, you can show that the volume of this solid is given by

$$V = \int_a^b A(x) \, dx.$$ 

For many of the exercises in the textbook, you will need to use techniques from elementary geometry to determine a formula for the cross sectional area function.

Consider the situation where the region between the graph of a continuous function $y = f(x)$, the $x$-axis, and the lines $x = a$ and $x = b$ is revolved about the $x$-axis. The resulting volume is called a solid of revolution. The cross sectional area at $x$ is a circle with radius $f(x)$. The cross sectional area is $A(x) = \pi f^2(x)$. The formula for the volume of this solid of revolution is

$$V = \pi \int_a^b f^2(x) \, dx.$$ 

This technique for finding the volume of a solid of revolution is called the disk method. It is useful to think of this formula as the integral of pi times the radius squared, the area of a circle.

When the region between two continuous functions, $f$ and $g$, with $0 \leq g(x) \leq f(x)$ and $a < x < b$ is revolved about the $x$-axis, the formula for the resulting solid of revolution is

$$V = \pi \int_a^b [f^2(x) - g^2(x)] \, dx.$$ 

Can you justify this formula? Can you explain why this technique is commonly called the washer method?

You need to be able to modify these formulas to handle other cases such as when a region is revolved around a horizontal line other than the $x$-axis. Often these methods can be modified to handle the case when the region is revolved about the $y$-axis. In the next section, we will discover another method that can often be used when it is difficult to apply the disk or washer method to a particular solid of revolution.
Suggested Homework:
Note: The assignment given by your instructor may be different.

Read Pages 420 – 433
Problems Available Online: Pages 429 – 433 – { 11, 13, 17, 19, 27, 29, 38, 62, 64 }
Problems from textbook: Pages 429 – 433 – { 45, 46, 53 }
Lesson 10
Volume by Shells
Section 6.4

Consider the situation where the region between the graph of a continuous function $y = f(x)$, the $x$-axis, and the lines $x = a$ and $x = b$, with $0 \leq a < b$, is revolved about the $y$-axis. To find the volume of this solid of revolution by the washer method can often be difficult. Instead of using disks, we look at cylinders.

![Diagram of a solid of revolution]

The cylinder will have a radius of $x$ and a height of $f(x)$. A careful examination of the proper Riemann sum will show that the volume can be obtained by integrating the areas of these cross sectional cylinders.

$$V = 2\pi \int_a^b xf(x) \, dx$$

This technique for finding the volume of a solid of revolution is called the shell method. It is useful to think of this formula as the integral of two pi times the radius times height, the area of a cylinder.

When the region between two continuous functions, $f$ and $g$, with $g(x) \leq f(x)$ and $0 \leq a < x < b$ is revolved about the $y$-axis, the formula for the resulting solid of revolution is

$$V = 2\pi \int_a^b x(f(x) - g(x)) \, dx$$

Can you justify this formula?
You need to be able to modify these formulas to handle other cases such as when a region is revolved around a vertical line other than the $y$-axis. Often the shell method can be modified to handle the case where the region is revolved around a horizontal line.

You need to be able to recognize which method, washer or shell, is appropriate for the problem at hand. Often both methods will be equally appropriate. You should make sure to do a few of these to see that you will get the same answer.

**Suggested Homework:**
Note: The assignment given by your instructor may be different.

Read Pages 434 – 445
Problems Available Online: Pages 442 – 445 – { 7, 15, 33, 35, 41, 49, 57 }  
Problems from textbook: Pages 442 – 4145 – { 13 }
Lesson 11

Arc Length

Section 6.5

Consider the problem of finding the length of the curve from the graph of a differentiable function \( y = f(x) \) for \( a < x < b \). This is the arc length of a curve. Let’s look at the length of a small section from \( x_{i-1} < x < x_i \). If we approximate this section with a straight line segment from the point \( (x_{i-1}, f(x_{i-1})) \) to \( (x_i, f(x_i)) \), we see that the length of this straight segment is

\[
\sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2}.
\]

Then summing these,

\[
\sum_{i=1}^n \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2} = \sum_{i=1}^n \sqrt{1 + \left(\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}\right)^2} (x_i - x_{i-1})
\]

\[
= \sum_{i=1}^n \sqrt{1 + \left(\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}\right)^2} (x_i - x_{i-1}).
\]

As usual let \( \Delta x_i = x_i - x_{i-1} \). By the Mean Value Theorem, there will be a point \( \bar{x}_i \) with \( x_{i-1} < \bar{x}_i < x_i \) where \( f'(\bar{x}_i) = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} \).

Therefore,

\[
\sum_{i=1}^n \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2} = \sum_{i=1}^n \sqrt{1 + [f'(\bar{x}_i)]^2} \Delta x_i.
\]

This is a Riemann sum from which we see that the arc length is given by,

\[
L = \int_a^b \sqrt{1 + [f'(x)]^2} \, dx.
\]

The Leibniz form of this formula is,

\[
L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx.
\]
It is useful to think of this formula as the integral of the hypotenuse of a triangle with leg lengths $dx$ and $dy$ where,

$$\sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx.$$ 

One algebra trick that you may find useful is

$$1 + \left(\frac{e^x - e^{-x}}{2}\right)^2 = 1 + \frac{1}{4} \left( e^{2x} - 2e^x e^{-x} + e^{-2x} \right)$$

$$= 1 + \frac{1}{4} e^{2x} - \frac{1}{2} + \frac{1}{4} e^{-2x}$$

$$= \frac{1}{4} e^{2x} + \frac{1}{2} + \frac{1}{4} e^{-2x}$$

$$= \frac{1}{4} \left( e^{2x} + 2e^x e^{-x} + e^{-2x} \right)$$

$$= \left( \frac{e^x + e^{-x}}{2} \right)^2$$

**Suggested Homework:**

Note: The assignment given by your instructor may be different.

Read Pages 445 – 451
Problems Available Online: Pages 450 – 451 – { 7, 11 }
Problems from textbook: Pages 450 – 451 – { 10, 31, 32 }
Lesson 12

Physical Applications

Section 6.7

This lesson is concerned with the application of integration to several problems from physics and engineering. We use Riemann sums to find the proper integral for the problem at hand. You should consider this section as a selection of examples on the use of Riemann sums as a strategy to solve these and similar problems.

The problems covered in this section include:

- finding the mass of a thin linear object that has a variable density.
- finding the work done when using a variable force to move an object along a straight line.
- finding the force exerted on a flat surface by a liquid.

In later courses, you may learn how to deal with physical problems that involve more than one dimension. An example of such a problem would be to find the work done when moving an object with variable force along a curve.

**Suggested Homework:**
Note: The assignment given by your instructor may be different.

Read Pages 459 – 471
Problems Available Online: Pages 467 – 471 – { 15, 22, 27, 39, 59 }
Problems from textbook: Pages 467 – 471 – { 32, 34, 49 }
Lesson 13
Logarithms and Exponential Functions
Section 6.8

In 1614, John Napier published an article, *Mirifici logarithmorum canonis descriptio* (A Description of the Wonderful Law of Logarithms.) Due to this article, John Napier has rightly been credited with the invention of logarithms. Napier spent over twenty years developing his theory before he published his results. His original purpose for this invention was to simplify the difficult multiplication and division problems associated with the development and use of trigonometry tables.

In modern terms, Napier essentially defined the logarithm to be a function whose derivative is a constant times the reciprocal of the variable. Napier did this by use of a geometric argument. A year later Napier and Henry Briggs collaborated to develop the common logarithm from the Napierian logarithm. For details, please see the sixth edition of the textbook, *An Introduction to the History of Mathematics* by Howard Eves.

As a point of local interest, the first Ph.D. in mathematics that was granted by Oregon State College (which became Oregon State University in 1961,) was to Howard Eves in 1948.

In this section, the natural logarithm will be defined as that function whose derivative is $f(x) = \frac{1}{x}$. The properties of the logarithm are then developed from this definition. The exponential function is defined as the inverse function to the logarithm.

View this section as an opportunity to review and gain a deeper understanding of the properties of logarithms and exponentials.

**Suggested Homework:**
Note: The assignment given by your instructor may be different.

Read Pages 471 – 481
Problems Available Online: Pages 480 – 481 – { 5, 7, 14, 16, 23, 27, 29, 35, 62 }
Lesson 14

Stock Formulas

and

Algebraic Manipulation

Section 7.1

With this lesson, we continue with our study of techniques of integration. The techniques of integration already covered are:

- Stock Formulas
- Algebraic Manipulation
- Substitution

In this section we will further explore these ideas. Carefully review the examples presented in the textbook. Often the techniques will come from trial and error along with basic algebraic techniques.

Here are some additional examples:

Split up fractions

\[ \int \frac{3x^2 + x}{\sqrt{x}} \, dx = \int \frac{3x^2}{x^{\frac{1}{2}}} + \frac{x}{x^{\frac{1}{2}}} \, dx \]

\[ = \int 3x^{\frac{3}{2}} + x^{\frac{1}{2}} \, dx \]

\[ = 2x^{\frac{5}{2}} + \frac{2}{3}x^{\frac{3}{2}} + C \]

Add zero

\[ \int \frac{x^2}{x + 4} \, dx = \int \frac{x^2 - 16 + 16}{x + 4} \, dx \]

\[ = \int \frac{x^2 - 16}{x + 4} \, dx + \int \frac{16}{x + 4} \, dx \]

\[ = \int \frac{(x - 4)(x + 4)}{x + 4} \, dx + \int \frac{16}{x + 4} \, dx \]

\[ = \int x - 4 \, dx + \int \frac{16}{x + 4} \, dx \]

\[ = \frac{1}{2}x^2 - 4x + 16\ln|x + 4| + C \]
Complete the square

\[
\int \frac{1}{x^2 + 6x + 13} \, dx = \int \frac{1}{x^2 + 6x + 9 - 4 + 13} \, dx \\
= \int \frac{1}{(x + 3)^2 + 4} \, dx \\
= \frac{1}{4} \int \frac{1}{(x + 3)^2 + 1} \, dx \\
= \frac{1}{4} \int \frac{1}{\left(\frac{x + 3}{2}\right)^2 + 1} \, dx \\
= \frac{1}{2} \int \frac{1}{u^2 + 1} \, du \\
= \frac{1}{2} \tan^{-1} u + C \\
= \frac{1}{2} \tan^{-1} \left( \frac{x + 3}{2} \right) + C
\]

Multiply by one

\[
\int \sqrt{\frac{1 + \cos \theta}{1 - \cos \theta}} \, d\theta = \int \sqrt{\frac{1 + \cos \theta(1 - \cos \theta)}{1 - \cos \theta(1 - \cos \theta)}} \, d\theta \\
= \int \sqrt{\frac{1 - \cos^2 \theta}{(1 - \cos \theta)^2}} \, d\theta \\
= \int \frac{|\sin \theta|}{1 - \cos \theta} \, d\theta \\
\text{NOTE: } \text{sgn}(x) = \begin{cases} 
1, & x \geq 0 \\
-1, & x < 0
\end{cases} \Rightarrow |x| = \text{sgn}(x)x
\]

\[
= \int \frac{\text{sgn}(\sin \theta) \sin \theta}{1 - \cos \theta} \, d\theta \\
u = 1 - \cos \theta \\
du = \sin \theta \, d\theta \\
\int \frac{du}{u} = \ln |u| + C \\
= \text{sgn}(\sin \theta) \ln |1 - \cos \theta| + C
\]

**Suggested Homework:**
Note: The assignment given by your instructor may be different.

Read Pages 511 – 516
Problems Available Online: Pages 514 – 516 – \{ 1, 3, 5, 11, 15, 23, 33, 38, 42 \}
Problems from textbook: Pages 514 – 516 – \{ 2, 4, 6, 30, 41 \}
Lesson 15
Integration by Parts
Section 7.2

The general principle in developing our techniques of integration is that every differentiation formula has a corresponding integration formula. From the product rule we see that

\[
\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)
\]

\[
\int \frac{d}{dx}(f(x)g(x)) \, dx = \int f'(x)g(x) \, dx + \int f(x)g'(x) \, dx
\]

\[
f(x)g(x) = \int f'(x)g(x) \, dx + \int f(x)g'(x) \, dx.
\]

Then

\[
\int f(x)g'(x) \, dx = f(x)g(x) - \int f'(x)g(x) \, dx.
\]

This is the integration by parts formula. It is usually seen and used in Leibniz notation.

\[
\begin{align*}
u &= f(x) & du &= f'(x) \, dx \\
v &= g(x) & dv &= g'(x) \, dx
\end{align*}
\]

After substitution this gives

\[
\int u \ dv = uv - \int v \ du.
\]

The basic idea is to convert a given difficult integral into a hopefully simpler integral.

Remember that once you make your choice for \( u \) then \( dv \) is everything else. As a general rule, your choice for \( dv \) should be something that you can easily integrate or your choice for \( u \) should be something that simplifies with differentiation. Your choice for \( u \) and \( dv \) is based on experience, practice, and trial and error.

We will do one example so you can see how to document your choices for \( u \) and \( dv \). The textbook has many examples of integration by parts covering the major twists and turns.
\[
\int x \ e^{-5x} \ dx = -\frac{1}{5} x e^{-5x} - \int \frac{e^{-5x}}{5} \ dx = -\frac{1}{5} x e^{-5x} + \frac{1}{5} \int e^{-5x} \ dx
\]

\[
u = x \quad dv = e^{-5x} \ dx
\]
\[
u = -\frac{e^{-5x}}{5}
\]

**Suggested Homework:**
Note: The assignment given by your instructor may be different.

Read Pages 516 – 523
Part 1
Problems Available Online: Pages 520 – 523 – \{ 1, 8, 10, 16, 19, 23, 24, 32, 42 \}
Problems from textbook: Pages 520 – 523 – \{ 33, 39 \}
Part 2
Problems Available Online: Pages 520 – 523 – \{ 48, 50, 57, 61, 70 \}
Problems from textbook: Pages 520 – 523 – \{ 44 \}
Lesson 16
Trigonometric Integrals

Section 7.3

Special techniques are shown for handling integrals of the form

\[ \int \sin^n x \cos^n x \, dx \text{ and } \int \tan^n x \sec^n x \, dx. \]

The techniques developed in this section can also be used for integrals of the form

\[ \int \cot^n x \csc^n x \, dx. \]

The following trigonometric identities are used in these techniques. You should learn these identities.

\begin{align*}
\cos^2 x + \sin^2 x &= 1 \\
\cos^2 x &= \frac{1}{2}(1 + \cos(2x)) \\
\sin^2 x &= \frac{1}{2}(1 - \cos(2x)) \\
\sin x \cos x &= \frac{1}{2}\sin(2x) \\
1 + \tan^2 x &= \sec^2 x \\
\cot^2 x + 1 &= \csc^2 x
\end{align*}

Suggested Homework:
Note: The assignment given by your instructor may be different.

Read Pages 523 – 531
Problems Available Online: Pages 529 – 531 – \{ 9, 14, 15, 16, 25, 28, 67 \}
Problems from textbook: Pages 529 – 531 – \{ 12, 21, 64 \}
The technique of integration covered in this section is informally called trig substitution. It is useful for integrals that contain terms of the forms:

- $a^2 + x^2$
- $a^2 - x^2$
- $x^2 - a^2$

Before you utilize a trig substitution, you should always carefully examine the integral to make sure that a simpler technique of integration cannot be used.

By convention, trig substitution is done by making use of either the sine, tangent, or secant functions. You should generally avoid using the cosine, cotangent, and cosecant functions as the functions of your substitution.

The following trig identities are often useful when converting back to your original variable after a trig substitution.

$$\sin(2x) = 2\sin x \cos x$$
$$\cos(2x) = 2\cos^2 x - 1$$

You often need to complete the square to put an integral into proper form before doing a trig substitution. As a quick review, recall that completing the square is based on the algebraic identity

$$\left(x + b\right)^2 = x^2 + 2bx + b^2.$$

Perhaps the following example of completing the square will spark some memories from your basic algebra course.
\[3x^2 + 8x + 7 = 3 \left( x^2 + \frac{8}{3}x \right) + 7\]

\[= 3 \left( x^2 + \frac{8}{3}x + \left( \frac{4}{3} \right)^2 - \left( \frac{4}{3} \right)^2 \right) + 7\]

\[\text{add } b^2 - b^2\]

\[= 3 \left( x + \frac{4}{3} \right)^2 - 3 \left( \frac{4}{3} \right)^2 + 7\]

\[\text{factor } x^2 + \frac{8}{3}x + \left( \frac{4}{3} \right)^2 = \left( x + \frac{4}{3} \right)^2\]

\[= 3 \left( x + \frac{4}{3} \right)^2 + \frac{5}{3}\]

\[\text{simplify}\]

**Suggested Homework:**

Note: The assignment given by your instructor may be different.

Read Pages 531 – 540
Problems Available Online: Pages 537 – 540 – \{ 1, 2, 3, 5, 13, 17, 28, 31, 33, 36, 44, 57, 62, 63, 71 \}
Problems from textbook: Pages 537 – 540 – \{ 4, 24 \}
Lesson 18  
Partial Fractions  
Section 7.5

A polynomial is a function that can be written in the form

\[ P(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_1 x + a_0, \]

where \( n \) is a nonnegative integer and \( a_n, a_{n-1}, a_{n-2}, \ldots, a_0 \) are constants. If \( a_n \neq 0 \), then \( n \) is the degree of the polynomial. As examples:

\[ P_1(x) = 3x^2 + 7 \text{ is a polynomial of degree } 2. \]
\[ P_2(x) = 3x^2 - 5x^4 + \sqrt{3}x - 19 \text{ is a polynomial of degree } 4. \]
\[ P_3(x) = (x + 1)(3x + 2)(x^2 + x + 1) \text{ is a polynomial of degree } 4. \]
\[ P_4(x) = 8 \text{ is a polynomial of degree } 0. \]

A rational function is a function that can be written as the quotient of two polynomials. The following functions are all examples of rational functions.

\[ f(x) = \frac{3x^2 + 7}{(2x + 3)(x - 7)(x^2 + 4)} \]
\[ g(x) = \frac{1}{x} \]
\[ h(x) = x^3 \]

Partial fractions decomposition is an algebraic technique of writing a rational function as a sum of simpler rational functions that can generally be more easily integrated.

The first step of a partial fractions decomposition is to make sure that the power of the numerator is strictly less than the power of the denominator. If needed, this is accomplished by long division. Hopefully, the following example will remind you of how to do long division with polynomials.

Consider the rational function, \( f(x) = \frac{3x^4 + 2x^3 + 8}{x^2 + 2x + 1} \). Since the power of the numerator is 4 and the power of the denominator is 2, the first step of a partial fractions decomposition will be a long division.
\[
\frac{3x^2 - 4x + 5}{x^2 + 2x + 1} \frac{3x^4 + 6x^3 + 3x^2}{3x^4 + 2x^3 + 0x^2 + 0x + 8}
\frac{-4x^3 - 3x^2 + 0x}{-4x^3 - 8x^2 - 4x}
\frac{5x^2 + 4x + 8}{5x^2 + 10x + 5}
\frac{-6x + 3}{-6x + 3}
\]

Then
\[
\frac{3x^4 + 2x^3 + 8}{x^2 + 2x + 1} = 3x^2 - 4x + 5 + \frac{3 - 6x}{x^2 + 2x + 1}.
\]

With partial fractions decomposition you will learn how to further decompose this to see that
\[
\frac{3x^4 + 2x^3 + 8}{x^2 + 2x + 1} = 3x^2 - 4x + 5 + \frac{9}{(x+1)^2} - \frac{6}{x+1}.
\]

Then
\[
\int \frac{3x^4 + 2x^3 + 8}{x^2 + 2x + 1} \, dx = \int 3x^2 - 4x + 5 + \frac{9}{(x+1)^2} - \frac{6}{x+1} \, dx.
\]

This second integral can now be handled quite easily.

It is often useful to complete the square to rewrite an irreducible quadratic, \( ax^2 + bx + c \), in the form \( a(x + \alpha)^2 + \beta \). For example,
\[
2x^2 + 12x + 27 = 2(x^2 + 6x) + 27
= 2(x^2 + 6x + 9 - 9) + 27
= 2(x^2 + 6x + 9) - 18 + 27
= 2(x + 3)^2 + 9.
\]

As a historical note, the first known use of integration by partial fractions was the following derivation of the integral of the secant by Isaac Barrow. Isaac Barrow (1630 – 1677) is best known as Isaac Newton’s teacher.
\[
\int \sec x \ dx = \int \frac{1}{\cos x} \ dx
\]
\[
= \int \frac{\cos x}{\cos^2 x} \ dx
\]
\[
= \int \frac{\cos x}{1 - \sin^2 x} \ dx \quad \quad u = \sin x \quad \quad du = \cos x \ dx
\]
\[
= \int \frac{1}{1 - u^2} \ dx \quad \quad \frac{1}{1-u^2} = \frac{1}{(1+u)(1-u)} = \frac{1}{2} \left( \frac{1}{1+u} + \frac{1}{1-u} \right)
\]
\[
= \int \frac{1}{2} \left( \frac{1}{1+u} + \frac{1}{1-u} \right) \ dx
\]
\[
= \frac{1}{2} \ln | 1+u | - \frac{1}{2} \ln | 1-u | + C
\]
\[
= \frac{1}{2} \ln \left| \frac{1+u}{1-u} \right| + C
\]
\[
= \frac{1}{2} \ln \left| \frac{1 + \sin x}{1 - \sin x} \right| + C
\]

The above derivation is a very loose translation of Isaac Barrow’s derivation. The integral notation and the modern concept of an indefinite integral did not yet exist in his time.

As a challenge, show that
\[
\frac{1}{2} \ln \left| \frac{1+\sin x}{1-\sin x} \right| = \ln \left| \sec x + \tan x \right|.
\]

**Suggested Homework:**
Note: The assignment given by your instructor may be different.

Read Pages 541 – 551
Problems Available Online: Pages 549 – 551 – \{ 13, 15, 17, 29, 46, 51, 66, 69, 79 \}
Problems from textbook: Pages 549 – 551 – \{ 25, 65, 76, 84 \}
Lesson 18
Integration Strategies
Section 7.5

The following steps may be useful in helping to determine which techniques of integration to use to evaluate a particular integral.

1. Is your integral one of the stock formulas?
2. Can the integrand be simplified or manipulated into a stock formula by algebraic manipulation?
3. Are there any obvious substitutions?
4. Can you classify the integral as to type?
   - Integration by Parts
   - Trigonometric Substitution
   - Partial Fractions
5. The last step is trial and error.
   - Try various algebraic manipulations.
   - Try various substitutions.
   - Try different choices for integration by parts.

At the end of this lesson are fifty integrals for practice. Following the problems are hints in case you are stuck. The solutions are included; don’t peek.

Other available strategies are to utilize tables of integrals, computer algebra systems, and numerical methods. Some numerical methods will be discussed in the next lesson.

There is a short table of integrals in the endpapers of your textbook. Larger tables of integrals are available. Often you will need to use your techniques of integration to convert your integral into one that matches one of the table entries.

Computer algebra systems are sophisticated software packages that can solve mathematical problems in symbolic form. There are many computer algebra systems available. The three that you are most likely to find available in a computer lab on campus are Mathcad, Maple, and Mathematica. Any of these three can do symbolic integration. Several handheld calculators contain computer algebra systems. Currently the most popular and available is the TI-89 Titanium made by Texas Instruments.
Some of the features of Mathematica are available at the Wolfram Alpha website, http://www.wolframalpha.com/. To give a test drive, enter integrate. You will then be presented with a box in which you can enter your function.

Be aware, that quite often an integral is best done by hand rather than machine. There still is no substitute for human intelligence. As an example, consider the following integration solution,

\[ \int \frac{1 + x}{1 - x} \, dx = \int \frac{\sqrt{1 + x}}{\sqrt{1 - x}} \left( \frac{\sqrt{1 + x}}{\sqrt{1 + x}} \right) \, dx \]

\[ = \int \frac{1 + x}{\sqrt{1 - x^2}} \, dx \]

\[ = \int \frac{1}{\sqrt{1 - x^2}} \, dx + \int \frac{x}{\sqrt{1 - x^2}} \, dx \]

\[ = \sin^{-1} x - \sqrt{1 - x^2} + C. \]

The TI-89 is unable to evaluate the integral in the given form. Wolfram Alpha evaluates this integral as

\[ \int \frac{1 + x}{\sqrt{1 - x^2}} \, dx = \frac{\left( \sqrt{x + 1} \right) \left( \sqrt{x + 1} (x - 1) + 2 \sqrt{1 - x} \sin^{-1} \left( \frac{\sqrt{x + 1}}{\sqrt{2}} \right) \right)}{\sqrt{x + 1}} + C. \]

The two answers are equal.

**Suggested Homework:**

Note: The assignment given by your instructor may be different.

Read Pages 551 – 557

Your instructor may assign a selection of integrals from the following problems.
The first 30 integrals do not require the use of either trigonometric substitution or partial fractions.

1. \( \int e^{5-4x} \, dx \)
2. \( \int \sin(\pi z) \, dz \)
3. \( \int \cos^2(3\theta) \, d\theta \)
4. \( \int xe^{-2x} \, dx \)
5. \( \int t^3 \ln t \, dt \)
6. \( \int \frac{4x}{\sqrt{9-4x^2}} \, dx \)
7. \( \int \frac{dx}{\sqrt{9-4x^2}} \)
8. \( \int \frac{e^z - e^{-z}}{e^z + e^{-z}} \, dt \)
9. \( \int \frac{\cos \alpha}{1 + \sin^2 \alpha} \, d\alpha \)
10. \( \int \frac{x}{1 + x^4} \, dx \)
11. \( \int x \tan^{-1} x \, dx \)
12. \( \int \sin^3 \varphi \, d\varphi \)
13. \( \int \sec^4 t \, dt \)
14. \( \int \frac{\tan(\sqrt{z})}{\sqrt{z}} \, dz \)
15. \( \int 5^{2x} \, dx \)
16. \( \int e^{3x} \sec(e^{3x}) \, dx \)
17. \( \int \frac{1}{1 + \sqrt{x}} \, dx \) 
18. \( \int y(2 - y)^3 \, dy \) 

19. \( \int e^{3x} \sin(2x) \, dx \) 
20. \( \int \frac{t^2 + 4t + 3}{t + 1} \, dt \) 

21. \( \int x^2 e^{5x} \, dx \) 
22. \( \int \tan^2(3\theta) \, d\theta \) 

23. \( \int \frac{d\alpha}{\sec(2\alpha)} \) 
24. \( \int (x-1)e^{-(x-1)^2} \, dx \) 

25. \( \int \tan^2 \theta \sec^2 \theta \, d\theta \) 
26. \( \int (w^2 + 2w + 1)\sqrt{w+1} \, dw \) 

27. \( \int \ln 7 \, dx \) 
28. \( \int \sec(2\theta) \, d\theta \) 

29. \( \int \ln(5x) \, dx \) 
30. \( \int \tan^5 \theta \sec^3 \theta \, d\theta \) 

The next 20 integrals may require either trigonometric substitution or partial fractions.

31. \( \int \frac{x + 3}{x^2 + 2x + 5} \, dx \) 
32. \( \int \frac{t^3}{t^2 + 1} \, dt \) 

33. \( \int \frac{1 + x}{\sqrt{1 - x}} \, dx \) 
34. \( \int \frac{dz}{z + \sqrt{z}} \)
35. \[ \int \frac{d\theta}{1 + \sin \theta} \]

36. \[ \int \frac{\sqrt{4 - x^2}}{x^2} \, dx \]

37. \[ \int \frac{x^2}{\sqrt{x^2 - 1}} \, dx \]

38. \[ \int \sqrt{9 - x^2} \, dx \]

39. \[ \int \frac{x^3}{x^2 - 2x + 8} \, dx \]

40. \[ \int \frac{x^2 + 5x}{(x-1)^3(x+1)} \, dx \]

41. \[ \int \frac{dt}{(t^2 - 6t + 13)^{\frac{3}{2}}} \, dt \]

42. \[ \int \frac{\cos \theta}{\sin^2 \theta + 2 \sin \theta + 2} \, d\theta \]

43. \[ \int \frac{x^3 + 3x^2 - 4x + 20}{x^3 - 16} \, dx \]

44. \[ \int \frac{dw}{\sqrt{w+1} - \sqrt{w}} \]

45. \[ \int \frac{\sqrt{x^2 - 2x + 2}}{x - 1} \, dx \]

46. \[ \int \frac{\sin \theta + \tan \theta}{\cos \theta} \, d\theta \]

47. \[ \int x \sin x \cos x \, dx \]

48. \[ \int \frac{\sqrt{x}}{4 - x} \, dx \]

49. \[ \int \frac{dx}{x^3 - x^2} \]

50. \[ \int \frac{dx}{(x^2 + 1)^2} \]
Hints for Practice Integrals

1. \( u = 5 - 4x \)

2. \( u = \pi z \)

3. \( \cos^2 x = \frac{1}{2} + \frac{1}{2} \cos(2x) \)

4. Parts

5. Parts

6. \( u = 9 - 4x^2 \)

7. \( \int \frac{dx}{\sqrt{1-x^2}} \) \( dx = \sin^{-1} x + C \)

8. \( u = e^t + e^{-t} \)

9. \( u = \sin \alpha \)

10. \( x^4 = (x^2)^2, \ u = x^2 \)

11. Parts

12. \( \sin^3 \varphi = \sin^2 \varphi \sin \varphi \)

13. \( \frac{d}{dt} \tan t = \sec^2 t \)

14. \( u = \sqrt{z} \)

15. \( \int a^u du = \frac{a^u}{\ln a} + C, \ a > 0, a \not= 1 \)

16. \( u = e^{3x} \)

17. \( u^2 = x \)

18. \( u = 2 - y \) or parts

19. Parts

20. \( t^2 + 4t + 3 = (t + 1)(t + 3) \)

21. Parts

22. \( 1 + \tan^2 x = \sec^2 x \)
23. $\sec x = \frac{1}{\cos x}$
24. $u = -(x - 1)^2$

25. $\frac{d}{dx} \tan x = \sec^2 x$
26. $x^2 + 2x + 1 = (x + 1)^2$

27. $\ln 7$ is a number
28. $u = 2\theta$

29. $\ln(5x) = \ln x + \ln 5$
30. $\frac{d}{dx} \sec x = \sec x \tan x$

31. $x^2 + 2x + 5 = \left(x^2 + 2x + 1\right) + 4$
32. Long division

33. $\sqrt{\frac{1 + x}{1 - x}} = \left(\frac{\sqrt{1 + x}}{\sqrt{1 - x}}\right) \sqrt{1 + x}$
34. $u^2 = z$

35. $\frac{1}{1 + \sin \theta} = \left(\frac{1}{1 + \sin \theta}\right) \frac{1 - \sin \theta}{1 - \sin \theta}$
36. $x = \sin \left(\frac{\theta}{2}\right)$

37. $x = \sec \theta$
38. $x = \sin \left(\frac{x}{3}\right)$

39. Long division
40. Partial fractions

41. $t^2 - 6t + 13 = \left(t^2 - 6t + 9\right) + 4$
42. $u = \sin \theta$
43. \( x^4 - 16 = (x^2 + 4)(x + 2)(x + 2) \)

44. Multiply by 1

45. \( x^2 - 2x + 2 = (x^2 - 2x + 1) + 1 \)

46. \( \tan \theta = \frac{\sin \theta}{\cos \theta} \)

47. Parts

48. \( u^2 = x \)

\[ u = x \quad dv = \sin x \cos x \, dx \]

49. \( x^3 - x^2 = x^2(x - 1) \)

50. \( x = \tan \theta \)
The first 30 integrals do not require the use of either trigonometric substitution or partial fractions.

1. \[ \int e^{5-4x} \, dx = -\frac{1}{4} e^{5-4x} + C \]

2. \[ \int \sin(\pi z) \, dz = -\frac{1}{\pi} \cos(\pi z) + C \]

3. \[ \int \cos^2(3\theta) \, d\theta = \frac{\theta}{2} + \frac{1}{12} \sin(6\theta) + C \]

4. \[ \int xe^{-2x} \, dx = \left( -\frac{x}{2} - \frac{1}{4} \right) e^{-2x} + C \]

5. \[ \int t^3 \ln t \, dt = \frac{t^4}{4} \ln t - \frac{1}{4} \int t^3 \, dt = \frac{t^4}{4} \ln t - \frac{1}{16} t^4 + C \]

6. \[ \int \frac{4x}{\sqrt{9-4x^2}} \, dx = -\sqrt{9-4x^2} + C \]

7. \[ \int \frac{dx}{\sqrt{9-4x^2}} = \frac{1}{2} \sin^{-1} \left( \frac{2x}{3} \right) + C \]

8. \[ \int \frac{e^t - e^{-t}}{e^t + e^{-t}} \, dt = \ln \left( e^t + e^{-t} \right) + C \]
9. \[ \int \frac{\cos \alpha}{1 + \sin^2 \alpha} \, d\alpha = \tan^{-1}(\sin \alpha) + C \]

10. \[ \int \frac{x}{1 + x^4} \, dx = \frac{1}{2} \int \frac{du}{1 + u^2} = \frac{1}{2} \tan^{-1} u + C = \frac{1}{2} \tan^{-1} \left( \frac{x}{2} \right) + C \]
    
    \[ u = x^2 \]
    
    \[ du = 2x \, dx \]

11. \[ \int x \tan^{-1} x \, dx \]
    
    \[ = \frac{1}{2} x^2 \tan^{-1} x - \frac{1}{2} \int \frac{x^2}{x^2 + 1} \, dx = \frac{1}{2} x^2 \tan^{-1} x - \frac{1}{2} \int 1 - \frac{1}{x^2 + 1} \, dx \]
    
    \[ = \frac{1}{2} (x^2 + 1) \tan^{-1} x - \frac{1}{2} x + C \]

12. \[ \int \sin^3 \varphi \, d\varphi = \int (1 - \cos^2 \varphi) \sin \varphi \, d\varphi = \frac{1}{3} \cos^3 \varphi - \cos \varphi + C \]

13. \[ \int \sec^4 t \, dt = \int (\tan^2 t + 1) \sec^2 t \, dx = \frac{1}{3} \tan^3 x + \tan x + C \]

14. \[ \int \frac{\tan \left( \sqrt{z} \right)}{\sqrt{z}} \, dz = 2 \ln \left| \sec \left( \sqrt{z} \right) \right| + C \]

15. \[ \int 5^{2x} \, dx = \frac{5^{2x}}{2 \ln 5} + C \]

16. \[ \int e^{3x} \sec(e^{3x}) \, dx = \frac{1}{3} \ln \left| \sec(e^{3x}) + \tan(e^{3x}) \right| + C \]
17.  \[
\int \frac{1}{1+\sqrt{x}} \, dx = \int 2u \, du = 2u - 2 \ln|1+u| + C = 2\sqrt{x} - 2 \ln(1 + \sqrt{x}) + C
\]

\[
u^2 = x
\]

\[
u = \sqrt{x}
\]

18.  \[
\int y(2 - y)^{\frac{3}{2}} \, dy = -\frac{2}{5}y(2 - y)^{\frac{5}{2}} + \frac{2}{5}\int (2 - y)^{\frac{5}{2}} \, dy = -\frac{2}{5}y(2 - y)^{\frac{5}{2}} - \frac{4}{35} (2 - y)^{\frac{7}{2}} + C
\]

19.  \[
\int e^{3x} \sin(2x) \, dx = \frac{3}{13} e^{3x} \sin(2x) - \frac{2}{13} e^{3x} \cos(2x) + C
\]

20.  \[
\int \frac{t^2 + 4t + 3}{t + 1} \, dt = \frac{1}{2}t^2 + 3t + C
\]

21.  \[
\int x^2 e^{5x} \, dx = \left(\frac{1}{5}x^2 - \frac{2}{25}x + \frac{2}{125}\right) e^{5x} + C
\]

22.  \[
\int \tan^2(3\theta) \, d\theta = \frac{1}{3} \tan(3\theta) - \theta + C
\]

23.  \[
\int \frac{d\alpha}{\sec(2\alpha)} = \int \cos(2\alpha) \, d\alpha = \frac{1}{2} \sin(2\alpha) + C
\]

24.  \[
\int (x - 1)e^{-(x-1)^2} \, dx = -\frac{1}{2} e^{-(x-1)^2} + C
\]
25. \[ \int \tan^2 \theta \sec^2 \theta \ d\theta = \frac{1}{3} \tan^3 \theta + C \]

26. \[ \int (w^2 + 2w + 1)\sqrt{w + 1} \ dw = \int (w + 1)^{\frac{1}{2}} (w + 1)^{\frac{5}{2}} \ dw = \int (w + 1)^{\frac{5}{2}} \ dw = \frac{2}{7} (w + 1)^{\frac{7}{2}} + C \]

27. \[ \int \ln 7 \ dx = x \ln 7 + C \]

28. \[ \int \sec(2\theta) \ d\theta = \frac{1}{2} \ln \left| \sec(2\theta) + \tan(2\theta) \right| + C \]

29. \[ \int \ln(5x) \ dx = x \ln(5x) - x + C \]

30. \[ \int \tan^5 \theta \sec^3 \theta \ d\theta \]

\[ = \int \tan^4 x \sec^2 x \sec x \tan x \ dx = \int \left( \sec^2 x - 1 \right)^2 \sec^2 x \sec x \tan x \ dx \]

\[ = \int \left( \sec^6 x - 2 \sec^4 x + \sec^2 x \right) \sec x \tan x \ dx \]

\[ = \frac{1}{7} \sec^7 x - \frac{2}{5} \sec^5 x + \frac{1}{3} \sec^3 x + C \]
The next 20 integrals may require either trigonometric substitution or partial fractions.

31. $$\int \frac{x + 3}{x^2 + 2x + 5} \, dx = \int \frac{x + 1 + 2}{(x + 1)^2 + 4} \, dx = \frac{1}{2} \ln(x^2 + 2x + 5) + \tan^{-1}\left(\frac{x + 1}{2}\right) + C$$

32. $$\int \frac{t^3}{t^2 + 1} \, dt = \int \left(t - \frac{t}{t^2 + 1}\right) \, dt = \frac{1}{2} t^2 - \frac{1}{2} \ln(t^2 + 1) + C$$

33. $$\int \frac{1 + x}{\sqrt{1 - x}} \, dx = \int \left(\frac{\sqrt{1 + x}}{\sqrt{1 - x}}\right) \left(\frac{\sqrt{1 + x}}{\sqrt{1 + x}}\right) \, dx = \int \frac{1 + x}{\sqrt{1 - x^2}} \, dx = \sin^{-1} x - \sqrt{1 - x^2} + C$$

34. $$\int \frac{dz}{z + \sqrt{z}} = \int \frac{2u}{u^2 + u} \, du = \int \frac{2}{u + 1} \, du = 2 \ln|u + 1| + C = 2 \ln\left|1 + \sqrt{z}\right| + C$$

$$u^2 = z$$
$$2udu = dz$$

35. $$\int \frac{d\theta}{1 + \sin \theta}$$

$$= \int \frac{1}{1 + \sin \theta} \left(\frac{1 - \sin \theta}{1 - \sin \theta}\right) \, d\theta = \int \frac{1 - \sin \theta}{1 - \sin^2 \theta} \, d\theta = \int \frac{1 - \sin \theta}{\cos^2 \theta} \, d\theta$$

$$= \int \sec^2 \theta - \sec \theta \tan \theta \, d\theta = \tan \theta - \sec \theta + C$$
36. \[ \int \frac{\sqrt{4-x^2}}{x^2} \, dx \quad x = 2 \sin \theta \]

\[ = \int \frac{4 \cos^2 \theta}{4 \sin^2 \theta} \, d\theta = \int \cot^2 \theta \, d\theta = \int \csc^2 \theta - 1 \, d\theta = -\cot \theta - \theta + C \]

\[ = -\frac{\sqrt{4-x^2}}{x} - \sin^{-1} \left( \frac{x}{2} \right) + C \]

37. \[ \int \frac{x^2}{\sqrt{x^2 - 1}} \, dx \quad x = \sec \theta \]

\[ = \int \sec^3 \theta \, d\theta = \frac{1}{2} \sec \theta \tan \theta + \frac{1}{2} \ln | \sec \theta + \tan \theta | + C \]

\[ = \frac{1}{2} x \sqrt{x^2 - 1} + \frac{1}{2} \ln | x + \sqrt{x^2 - 1} | + C \]

38. \[ \int \sqrt{9-x^2} \, dx \quad x = 3 \sin \theta \]

\[ = \int 9 \cos^2 \theta \, d\theta = \frac{9}{2} \theta + \frac{9}{2} \sin \theta \cos \theta + C = \frac{9}{2} \sin^{-1} \left( \frac{x}{3} \right) + \frac{1}{2} x \sqrt{9-x^2} + C \]

39. \[ \int \frac{x^3}{x^2 - 2x + 8} \, dx \]

\[ = \int x + 2 - \frac{4x + 16}{x^2 - 2x + 8} \, dx = \frac{1}{2} x^2 + 2x - 4 \int \frac{x - 1 + 5}{(x - 1)^2 + 7} \, dx \]

\[ = \frac{1}{2} x^2 + 2x - 2 \ln | (x - 1)^2 + 7 | - \frac{20}{\sqrt{7}} \tan^{-1} \left( \frac{x - 1}{\sqrt{7}} \right) + C \]
40. \[ \int \frac{x^3 + 5x}{(x-1)^3(x+1)} \, dx \]

\[ = \int \frac{3}{(x-1)^2} + \frac{2}{x-1} - \frac{1}{x+1} \, dx = -\frac{3}{x-1} + 2 \ln |x-1| - \ln |x+1| + C \]

41. \[ \int \frac{dt}{(t^2 - 6t + 13)^{\frac{3}{2}}} \quad t = 2 \tan(t - 3) \]

\[ = \int \frac{dt}{(t^2 + 4)^{\frac{3}{2}}} = \int \frac{2 \sec^2 \theta}{8 \sec^3 \theta} \, d\theta = \frac{1}{4} \int \cos \theta \, d\theta = \frac{1}{4} \sin \theta + C = \frac{t - 3}{4\sqrt{t^2 - 6t + 13}} + C \]

42. \[ \int \frac{\cos \theta}{\sin^2 \theta + 2 \sin \theta + 2} \, d\theta \quad u = \sin \theta \]

\[ du = \cos \theta \, d\theta \]

\[ = \int \frac{du}{u^2 + 2u + 2} = \int \frac{du}{(u + 1)^2 + 1} = \tan^{-1}(u + 1) + C = \tan^{-1}(1 + \cos \theta) + C \]

43. \[ \int \frac{x^3 + 3x^2 - 4x + 20}{x^4 - 16} \, dx \]

\[ = \int \frac{x - 1}{x^2 + 4} - \frac{1}{x+2} + \frac{1}{x-2} \, dx = \frac{1}{2} \ln(x^2 + 4) - \frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right) - \ln |x + 2| + \ln |x - 2| + C \]
44. \[ \int \frac{dw}{\sqrt{w+1} - \sqrt{w}} \]

\[ = \int \left( \frac{1}{\sqrt{w+1} - \sqrt{w}} \right) \left( \frac{\sqrt{w+1} + \sqrt{w}}{\sqrt{w+1} + \sqrt{w}} \right) \, dw = \int \frac{\sqrt{w+1} + \sqrt{w}}{w+1} \, dw = \frac{2}{3} (w+1)^{\frac{3}{2}} + \frac{2}{3} w^{\frac{3}{2}} + C \]

45. \[ \int \frac{x^2 - 2x + 2}{x-1} \, dx \quad x-1 = \tan \theta \]

\[ = \int \frac{\sqrt{(x-1)^2 + 1}}{x-1} \, dx = \int \sec^3 \theta \, d\theta = \int \frac{(\tan^2 \theta + 1) \sec \theta}{\tan \theta} \, d\theta = \int \sec \theta \tan \theta + \frac{\sec \theta}{\tan \theta} \, d\theta \]

\[ = \sec \theta + \ln | \csc \theta + \cot \theta | + C \]

\[ = \sqrt{x^2 - 2x + 2} - \ln \left| \frac{\sqrt{x^2 - 2x + 2} + 1}{x-1} \right| + C \]

46. \[ \int \frac{\sin \theta + \tan \theta}{\cos \theta} \, d\theta \]

\[ = \int \frac{\sin \theta + \sec \theta \tan \theta}{\cos \theta} \, d\theta = \int \tan \theta + \sec \theta \tan \theta \, d\theta = \ln | \sec \theta | + \sec \theta + C \]

47. \[ \int x \sin x \cos x \, dx = \frac{1}{2} x \sin^2 x - \frac{1}{2} \int \sin^2 x \, dx = \frac{1}{2} x \sin^2 x - \frac{1}{4} x + \frac{1}{4} \sin x \cos x + C \]

48. \[ \int \frac{\sqrt{x}}{4-x} \, dx \quad u^2 = x \]

\[ = \int \frac{2u^2}{4-u^2} \, du = \int \frac{2}{u+2} - \frac{2}{u-2} \, du = -2u + 2 \ln | u+2 | - 2 \ln | u-2 | + C \]

\[ = -2\sqrt{x} + 2 \ln | \sqrt{x} + 2 | - 2 \ln | \sqrt{x} - 2 | + C \]
49. \[ \int \frac{dx}{x^3 - x^2} = \int \frac{1}{x-1} - \frac{1}{x^2} - \frac{1}{x} \, dx = \ln |x - 1| + \frac{1}{x} - \ln |x| + C \]

50. \[ \int \frac{dx}{(x^2 + 1)^2} \quad x = \tan \theta \quad dx = \sec^2 \theta \]

\[ = \int \frac{\sec^2 \theta}{\sec^4 \theta} \, d\theta = \int \cos^2 \theta \, d\theta = \frac{1}{2} \theta + \frac{1}{2} \sin \theta \cos \theta + C = \frac{1}{2} \tan^{-1} x + \frac{x}{2(x^2 + 1)} + C \]
Lesson 20
Numerical Integration
Section 7.7

Many elementary functions do not have an antiderivative that can be expressed in terms of elementary functions. Nevertheless, it is often necessary to be able to compute definite integrals involving these functions.

Examples of functions known to not have an antiderivative that can be expressed in terms of elementary functions are

\[ f(x) = e^{-x^2} \]
\[ g(x) = \frac{\sin x}{x} \]
\[ h(x) = \sqrt{1 + x^3}. \]

In this section, three numerical methods for finding approximations of definite integrals will be covered. The Midpoint Rule is a Riemann sum where the midpoint of the interval is used. The Trapezoid Rule uses trapezoids instead of rectangles to approximate the integral. Simpson’s Rule uses the areas under parabolas.

A numerical approximation is worthless if you cannot say anything about the error of the approximation. For both the Midpoint Rule and the Trapezoid Rule the approximation is exact if you are integrating a linear function. The error comes from the convexity of the curve for these two methods. As the absolute value of the second derivative gets larger, you can expect the error of the approximation to increase. Simpson’s Rule gives an exact value if you are approximating the area under a parabola. For Simpson’s Rule, the larger the value of the absolute value of the fourth derivative, the larger you can expect the error of your approximation. For all methods, you can generally increase your accuracy by increasing the number of sampling points. However, you need to be careful here. By increasing the number of sampling points, you are also increasing the number of needed calculations. This increases the error due to the round-off errors in your calculations.

**Suggested Homework:**
Note: The assignment given by your instructor may be different.

Read Pages 557 – 569
Problems Available Online: Pages 566 – 569 – \{ 1, 7, 26, 39, 42, 43 \}
Lesson 21

Improper Integrals

Section 7.8

An integral is improper if

- one or both of the limits of integration involve plus or minus infinity, or
- the integral is over or up to a vertical asymptote in the integrand.

Examples of improper integrals are

\[
\begin{align*}
\int_0^\infty e^{-3x} \, dx \\
\int_{-\infty}^\infty \frac{dx}{1 + x^2} \\
\int_4^{10} \frac{dx}{4 - x} \\
\int_{-2}^2 \frac{dx}{\sqrt[3]{x}}
\end{align*}
\]

Improper integrals are defined and evaluated by using limits.

Suggested Homework:
Note: The assignment given by your instructor may be different.

Read Pages 570 – 581
Problems Available Online: Pages 578 – 581 – \{ 4, 5, 9, 17, 32, 35, 36, 62, 64, 74, 75 \}
Problems from textbook: Pages 578 – 581 – \{ 18, 61 \}
Lesson 22
Exponential Models
Section 6.9

Many mathematical models begin with the assumption that the rate of change of the quantity of the substance you are modeling is proportional to the amount of substance present. In mathematical terms this says that if \( y(t) \) is the amount of substance present at time \( t \), then

\[
\frac{dy}{dt} = ky, 
\]

where \( k \) is a proportionality constant. This leads to the equation

\[
y(t) = Ce^{kt}
\]

where \( C \) is a constant.

If \( k > 0 \), this is called an exponential growth model. For \( k < 0 \), we have an exponential decay model.

In this section, we will see many examples where these models can be applied. We will also explore the concepts of doubling time and half-life.

**Suggested Homework:**
Note: The assignment given by your instructor may be different.

Read Pages 482 – 490
Problems Available Online: Pages 488 – 490 – \{ 11, 27, 29, 31, 32 \}
Problems from textbook: Pages 488 – 490 – \{ 28 \}
Lesson 23

Introduction to Differential Equations

Section 7.9

A differential equation is an equation that involves an unknown function \( y \) and this function’s derivative or derivatives. Examples of differential equations are

\[
\frac{dy}{dx} = x^2
\]
\[
\frac{dy}{dx} = -0.2y
\]
\[
\frac{dy}{dx} = -0.2xy
\]
\[
\frac{d^2y}{dx^2} + 4y = \sin(2x)
\]

The order of a differential equation is the order of the highest-order derivative. The last example in the above is an example of a second-order differential equation.

Many mathematical models are based on statements or laws regarding the rates at which certain quantities change. You’ve already seen some examples of this in the section on exponential growth and decay. The mathematical equations used to describe these models generally will involve differential equations. A good argument can be made that one of the primary reasons for studying calculus is to facilitate the study of differential equations.

A first-order differential equation of the form, \( \frac{dy}{dx} = f(x,y) \), is separable if you can factor \( f(x,y) = \frac{h(x)}{g(y)} \). The first three examples in the list above are separable differential equations. To solve a separable differential equation you separate and integrate.

\[
\int g(y) \frac{dy}{dx} \, dx = \int h(x) \, dx
\]
\[
\int g(y) \, dy = \int h(x) \, dx
\]
In this section, you will have the opportunity to practice solving separable differential equations. A first order differential equation with an initial condition is a problem of the form

\[ \frac{dy}{dx} = f(x,y), \quad y(t_0) = y_0, \]

where \( t_0 \) and \( y_0 \) are fixed values. The initial condition is used to determine the value of the constant of integration. Each initial condition will give a solution curve.

The direction field of a first order differential equation \( \frac{dy}{dx} = f(x,y) \) is a graph where the slope of a solution curve is shown at a selection of points. A direction field is useful in giving a visual representation of the solution curves.

Below is a direction field for the differential equation \( \frac{dy}{dx} = -0.2xy \). Below that is the same direction field with the solution curve plotted that corresponds to the initial condition \( y(0) = 6 \).
This differential equation is separable.

\[
\frac{dy}{dx} = -0.2xy
\]

\[
\frac{1}{y} \frac{dy}{dx} = -0.2x
\]

\[
\int \frac{dy}{y} = \int -0.2x \, dx
\]

\[
\ln|y| = -0.1x^2 + C
\]

\[
|y| = e^{-0.1x^2 + C}
\]

\[
|y| = e^C e^{-0.1x^2}
\]

From the direction field you can see that if \( y \) is ever positive, then it is always positive. Similarly if \( y \) is ever negative, then it is always negative. This can be verified without the use of the direction field.

When we apply the initial condition, \( y(0) = 6 \), the solution \( y = 6e^{-0.1x^2} \) is obtained.
**Suggested Homework:**
Note: The assignment given by your instructor may be different.

Read Pages 581 – 592
Problems Available Online: Pages 589 – 592 – { 21, 31, 43, 44, 45 }
Problems from textbook: Pages 589 – 592 – { 17, 25, 49 }
A selection of laboratory activities are found in this section. Your instructor will inform you which activities have been assigned for your particular section of the course.

**Overview of Laboratory Activities:**
For each assigned laboratory activity, your graduate teaching assistant will have you work in small groups. You will need to write up a laboratory report following the format given for each lab. The reports are due the week following the laboratory activity. Your instructor and your graduate teaching assistant will provide more information about what is expected and about the grading policy for the laboratory activities.

**Procedure:**
Work on the problems during the laboratory with your group. For most of the labs do your work during class on separate paper, not on the actual pages describing the activity. Finish the work on these pages during the next week. Write a legible report with full justification for your work and conclusions. Hand in your report to your teaching assistant. Your work will not be graded if it is written illegibly and is not organized.

**Acknowledgments:**
Some of these labs were created by the author of this study guide and by Dianne Hart. Other labs have been taken from previous Mth 25 study guides.
Laboratory: Differentiation Review and Antiderivatives

This lab is designed to be done as a group project with each group consisting of 3 to 5 students.

Part 1: Differentiation Review

1. Read Part 2: Antiderivatives for instruction on how to make flash cards. Use the table on the next three pages to create flash cards for the basic formulas for differentiation. Fill any spaces left with basic derivatives that you should be able to do without the use of pen and paper.

2. The following problems should be done independently and not as a group. Afterwards compare your answers with your group. Discuss and correct discrepancies. The groups write-up should consist of a separate set of solutions from each member of the group.

Evaluate and simplify the following derivatives:

1. \( \frac{d}{dx}(5x^3 + 8x + \sqrt{7}) \)
2. \( \frac{d}{dx}(3x\sqrt{x^3 + 5}) \)
3. \( \frac{d}{dt}(4t^3 \cos(5t)) \)
4. \( \frac{d}{dx}(e^{x^3} \tan(x)) \)
5. \( \frac{d}{dz}\left(\frac{3z^2 + 4}{z^3 + z}\right) \)
6. \( \frac{d}{d\theta}\left(\frac{\sin(3\theta)}{4 + \cos(3\theta)}\right) \)
7. \( \frac{d}{dt}(\sqrt{t} \ln(5t^2)) \)
8. \( \frac{d}{dx}\left(\sin^{-1}(x^2)\right) \)
9. \( \frac{d}{dx}(x^5) \)
10. \( \frac{d}{ds}\left(\frac{\tan^{-1}s}{s}\right) \)
11. \( \frac{d}{dx}(x^2 e^{x^2 + 5}) \)
12. \( \frac{d}{dw}\left(\ln(\sec^2(3w))\right) \)
13. \( \frac{d}{dx}(x^2 e^{-x} \sin(\pi x)) \)
14. \( \frac{d}{dy}\left(\sqrt{\frac{y^3 + 4^3}{y + 1}}\right) \)
15. \( \frac{d}{dx}(x^\cos(x)) \)
Part 2: Antiderivatives

The following 9 pages can be printed and used as flash cards. Cut along the solid lines and fold on the dotted line. You can also cut on the dotted line and tape to 3 by 5 index cards.

Make the flash cards and use them with your group to study and learn these formulas.

There are 51 formulas on these cards. It would be nice to have an even 60 formulas. Make up 9 more antiderivative problems that you should be able to solve without the use of pen and paper. Use the following table for your write-up.

<p>| | |</p>
<table>
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<tbody>
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<td>52.</td>
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<td>60.</td>
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<tr>
<td>1. ( \int x^p , dx ), ( p \neq -1 )</td>
<td>1. ( \frac{x^{p+1}}{p+1} + C )</td>
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<tr>
<td>2. ( \int \frac{1}{x} , dx )</td>
<td>2. ( \ln</td>
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<tr>
<td>3. ( \int e^{kx} , dx )</td>
<td>3. ( \frac{1}{k} e^{kx} + C )</td>
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<tr>
<td>4. ( \int \ln(x) , dx )</td>
<td>4. ( x \ln(x) - x + C )</td>
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<tr>
<td>5. ( \int \sin(kx) , dx )</td>
<td>5. ( -\frac{1}{k} \cos(kx) + C )</td>
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<tr>
<td>6. ( \int \cos(kx) , dx )</td>
<td>6. ( \frac{1}{k} \sin(kx) + C )</td>
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<tr>
<td></td>
<td>[ \int \tan(kx) , dx ]</td>
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<td>7</td>
<td>[ \int \cot(kx) , dx ]</td>
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<td>[ \int \sec(kx) , dx ]</td>
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<td>[ \int \csc(kx) , dx ]</td>
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<td>10</td>
<td>[ \int \sec^2(kx) , dx ]</td>
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<td>11</td>
<td>[ \int \csc^2(kx) , dx ]</td>
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<tr>
<td>13. $\int \sec(kx)\tan(kx) , dx$</td>
<td>13. $\frac{1}{k}\sec(kx) + C$</td>
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<tr>
<td>14. $\int \csc(kx)\cot(kx) , dx$</td>
<td>14. $-\frac{1}{k}\csc(kx) + C$</td>
</tr>
<tr>
<td>15. $\int \frac{dx}{x^2 + k^2}, \quad k &gt; 0$</td>
<td>15. $\frac{1}{k}\tan^{-1}\left(\frac{x}{k}\right) + C$</td>
</tr>
<tr>
<td>16. $\int \frac{dx}{\sqrt{k^2 - x^2}}, \quad k &gt; 0$</td>
<td>16. $\sin^{-1}\left(\frac{x}{k}\right) + C$</td>
</tr>
<tr>
<td>17. $\int \frac{dx}{x\sqrt{x^2 - k^2}}, \quad k &gt; 0$</td>
<td>17. $\frac{1}{k}\sec^{-1}\left</td>
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<tr>
<td>18. $\int e^x , dx$</td>
<td>18. $e^x + C$</td>
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<td>No.</td>
<td>Integral</td>
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<td>19.</td>
<td>$\int \sin(x) , dx$</td>
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<tr>
<td>20.</td>
<td>$\int \cos(x) , dx$</td>
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<td>21.</td>
<td>$\int \tan(x) , dx$</td>
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<td>22.</td>
<td>$\int \cot(x) , dx$</td>
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<td>23.</td>
<td>$\int \sec(x) , dx$</td>
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<td>24.</td>
<td>$\int \csc(x) , dx$</td>
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<td>Equation</td>
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<td>25.</td>
<td>$\int \sec^2(x) , dx$</td>
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<tr>
<td>26.</td>
<td>$\int \csc^2(x) , dx$</td>
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<tr>
<td>27.</td>
<td>$\int \sec(x)\tan(x) , dx$</td>
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<td>28.</td>
<td>$\int \csc(x)\cot(x) , dx$</td>
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<td>29.</td>
<td>$\int \frac{dx}{x^2 + 1}$</td>
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<td>30.</td>
<td>$\int \frac{dx}{\sqrt{1-x^2}}$</td>
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<td>31.</td>
<td>[ \int \frac{dx}{x\sqrt{x^2 - 1}} ]</td>
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<td>= \sec^{-1}</td>
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<td>32.</td>
<td>[ \int (3t + 5) , dt ]</td>
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<tr>
<td></td>
<td>= \frac{3}{2}t^2 + 5t + C</td>
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<tr>
<td>33.</td>
<td>[ \int \frac{dz}{z^3} ]</td>
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<td></td>
<td>= \int z^{-3} , dz = -\frac{1}{2}z^{-2} + C</td>
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<tr>
<td>34.</td>
<td>[ \int 5x^{-1} , dx ]</td>
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<td>= 5\ln</td>
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<tr>
<td>35.</td>
<td>[ \int \sqrt{x} , dx ]</td>
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<tr>
<td></td>
<td>= \int x^{\frac{1}{2}} , dx = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{2}{3}x^{\frac{3}{2}} + C</td>
</tr>
<tr>
<td>36.</td>
<td>[ \int t^7 , dt ]</td>
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<tr>
<td></td>
<td>= \frac{t^8}{8} + C</td>
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<tr>
<td>Problem</td>
<td>Integral</td>
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<td>----------</td>
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<tr>
<td>37.</td>
<td>$\int e^{-x} , dx$</td>
</tr>
<tr>
<td>38.</td>
<td>$\int e^{3\theta} , d\theta$</td>
</tr>
<tr>
<td>39.</td>
<td>$\int \sin(5\theta) , d\theta$</td>
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<tr>
<td>40.</td>
<td>$\int \cos(4\beta) , d\beta$</td>
</tr>
<tr>
<td>41.</td>
<td>$\int \tan(6x) , dx$</td>
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<tr>
<td>42.</td>
<td>$\int \cot(\pi x) , dx$</td>
</tr>
<tr>
<td>Question</td>
<td>Answer</td>
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<td>----------</td>
<td>--------</td>
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<tr>
<td>43. ( \int \sec(\pi \theta) , d\theta )</td>
<td>( = \frac{1}{\pi} \ln</td>
</tr>
<tr>
<td>44. ( \int \csc(3x) , dx )</td>
<td>( = -\frac{1}{3} \ln</td>
</tr>
<tr>
<td>45. ( \int \sec^2(\pi x) , dx )</td>
<td>( = \frac{1}{\pi} \tan(\pi x) + C )</td>
</tr>
<tr>
<td>46. ( \int \csc^2(3t) , dt )</td>
<td>( = -\frac{1}{3} \cot(3t) + C )</td>
</tr>
<tr>
<td>47. ( \int \sec(5\alpha)\tan(5\alpha) , d\alpha )</td>
<td>( = \frac{1}{5} \sec(5\alpha) + C )</td>
</tr>
<tr>
<td>48. ( \int \csc(2x)\cot(2x) , dx )</td>
<td>( = -\frac{1}{2} \csc(2x) + C )</td>
</tr>
<tr>
<td>49. ( \int \frac{dx}{x^2 + 25} )</td>
<td>49. ( = \frac{1}{5} \tan^{-1} \left( \frac{x}{5} \right) + C )</td>
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<tr>
<td>50. ( \int \frac{dx}{\sqrt{4 - x^2}} )</td>
<td>50. ( = \sin^{-1} \left( \frac{x}{2} \right) + C )</td>
</tr>
<tr>
<td>51. ( \int \frac{dx}{x\sqrt{x^2 - 9}}, ; k &gt; 0 )</td>
<td>51. ( = \frac{1}{3} \sec^{-1} \left</td>
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</tbody>
</table>
Mth 252 Lab: The Fundamental Theorem of Calculus

The purpose of this lab is to deepen your understanding of antiderivatives presented in integral form as area functions. This will require that you recall how to glean information about a function based on the graph of its derivative, i.e., you will need to recall some basic Mth 251 concepts.

The graph of \( f \) illustrated below is made up of line segments and two semicircles (of radius 1).

Let \( A(x) = \int_{-4}^{x} f(t) \, dt \) and \( F(x) = \int_{-1}^{x} f(t) \, dt \) be two area functions for \( f \). (Before you start, you need to discuss the relationship between the function \( f \) and the functions \( A \) and \( F \).)

Do all work on another sheet of paper.

1. Determine the intervals over which \( A \) and \( F \) are increasing and those over which \( A \) and \( F \) are decreasing. Discuss how you are using the graph of \( f \) to determine this, referring to concepts from Mth 251.

2. Determine the intervals over which the graphs of \( A \) and \( F \) are concave up and those over which the graphs of \( A \) and \( F \) are concave down. Discuss how you are using the graph of \( f \) to determine this, referring to concepts from Mth 251.

3. State the \( x \) coordinates of any local maxima and local minima for \( A \) and \( F \). Discuss how you are using the graph of \( f \) to determine this, referring to concepts from Mth 251.

4. State the \( x \) coordinates of any inflection points for \( A \) and \( F \). Discuss how you are using the graph of \( f \) to determine this, referring to concepts from Mth 251.

5. Use the formula given for \( A \) and the graph of \( f \) to complete the table below. As you find each value, write down the appropriate integral expression for \( A \) and then show/explain how you are finding each value. Use geometric formulas to evaluate any definite integrals. First, find the exact value and then round to one decimal place.

<table>
<thead>
<tr>
<th>( x )</th>
<th>-4</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A(x) )</td>
<td>[</td>
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<td></td>
<td></td>
<td></td>
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</table>
6. Use the formula given for \( F \) and the graph of \( f \) to complete the table below. As you find each value, write down the appropriate integral expression for \( F \) and then show/explain how you are finding each value. Use geometric formulas to evaluate any definite integrals. First, find the exact value and then round to one decimal place.

<table>
<thead>
<tr>
<th>( x )</th>
<th>(-4)</th>
<th>(-3)</th>
<th>(-2)</th>
<th>(-1)</th>
<th>(0)</th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
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<tbody>
<tr>
<td>( F(x) )</td>
<td></td>
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<td></td>
<td></td>
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<td></td>
</tr>
</tbody>
</table>

7. Plot all the points found in #5. Sketch the graph of \( A \) carefully noting the increasing/decreasing behavior and the concavity. Label any local maxima as \( M \), any local minima as \( m \), and any inflection points as \( ip \).

8. On the same axes, plot all the points found in #6. Sketch the graph of \( F \) carefully noting the increasing/decreasing behavior and the concavity. Label any local maxima as \( M \), any local minima as \( m \), and any inflection points as \( ip \).

9. Describe what you notice about the graphs of \( A \) and \( F \). Does this support what you think you should happen with regard to the graphs of the functions \( A \) and \( F \)? Why or why not?

10. Lastly, symbolically, find

\[
A'(x) = \frac{d}{dx} \int_{-4}^{x} f(t) \, dt = \text{__________} \\
F'(x) = \frac{d}{dx} \int_{-1}^{x} f(t) \, dt = \text{__________}.
\]

Be careful that your answer is written with the appropriate variable.

11. Extra thought. We see that \( A(-4) = 0 \). Can you think of a way to easily write another antiderivative of \( f \) where \( A(-4) = 5 \)? This should require no work – only some thought.
Mth 252 Lab: The Substitution Rule (Change of Variables)

The purpose of this lab is to practice using The Substitution Rule. It will probably be most helpful to you to work each of these by yourself and then discuss them in your groups.

Evaluate the following indefinite integrals. Show the change of variables used for each problem. Check your work by differentiation.

1. \( \int 2x(x^2 + 1)^4 \, dx \)  
2. \( \int 8x \cos(4x^2 + 3) \, dx \)  
3. \( \int \sin^3 x \cos x \, dx \)  
4. \( \int \frac{6x + 1}{\sqrt{3x^2 + x}} \, dx \)  
5. \( \int \left( \frac{\sqrt{x} + 1}{2\sqrt{x}} \right)^4 \, dx \)  
6. \( \int \sec^2(5x) \, dx \)  
7. \( \int \frac{e^{2x}}{e^{2x} + 1} \, dx \)  
8. \( \int \frac{\sin x}{\cos^4 x} \, dx \)  
9. \( \int x^2 e^{x^3 + 1} \, dx \)  
10. \( \int \frac{4}{\sqrt{1 - 9x^2}} \, dx \)  
11. \( \int \frac{1}{x(\ln x + 3)^5} \, dx \)  
12. \( \int \frac{\sec^2 x}{\tan x + 5} \, dx \)  
13. \( \int (x^{3/2} + 8)^5 \sqrt{x} \, dx \)  
14. \( \int \frac{5}{3x - 1} \, dx \)  
15. \( \int \sin(5x) \, dx \)  
16. \( \int e^{-6x} \, dx \)  
17. \( \int \frac{5}{(6x - 1)^4} \, dx \)  
18. \( \int \frac{1}{16x^2 + 1} \, dx \)  
19. \( \int \left( \ln x + 2x^3 - 5 \right)^{-10} \left( \frac{1}{x^2} + 6x \right) \, dx \)  
20. \( \int (\sin \theta + 6)e^{\cos \theta - 6\theta} \, d\theta \)  
21. \( \int \tan^7 \theta \sec^2 \theta \, d\theta \)  
22. \( \int \sin(\tan z - \pi^6) \sec^2 z \, dz \)  
23. \( \int \frac{(\arctan x)^4}{1 + x^2} \, dx \)  
24. \( \int \frac{6x^2}{2x^3 + 5} \, dx \)  

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Mth 252 Lab
Regions Between Curves

From the textbook: Section 6.2, problems 55, 65, and 67.
Math 252 Lab  
Volume of a Solid of Revolution

1. In class we developed the formula for the volume of a solid that is obtained when the region under the graph of \( y = f(x) \) and above the graph of \( y = g(x) \) from \( a \) to \( b \) with \( f(x) \geq g(x) \) and \( 0 \leq a \leq x < b \) is revolved about the y-axis. Using this formula is called the Shell Method.

\[ V = \int_a^b 2\pi x (f(x) - g(x)) \, dx \]

Reproduce your class notes showing the Riemann sum development of this formula. Clearly explain each step.

2. Take the region enclosed by the graphs \( y = \sqrt{1-x} \), \( y = 0 \) and \( x = 0 \).
   (a) Sketch the graph of this region.
   (b) Use the shell method to find the volume of the solid of revolution when this region is revolved about the y-axis.
   (c) Use the disk method to find the volume of the solid of revolution when this region is revolved about the y-axis. You should get the same answer as in part (b).

3. Take the region under the graph \( y = \cos x \) and above the graph \( y = \sin x \) for \( 0 \leq x \leq \frac{\pi}{4} \).
   (a) Sketch the graph of this region.
   (b) Find the volume of the solid of revolution that is generated when this region is rotated about the x-axis.
   (c) Find the volume of the solid of revolution that is generated when this region is rotated about the line \( y = 2 \).
   (d) Set up the integral you will need to evaluate to find the volume of the solid revolution that is generated when this region is rotated about the y-axis.
   (e) Set up the integral you will need to evaluate to find the volume of the solid revolution that is generated when this region is rotated about the line \( x = -3 \).

NOTE: Once we cover integration by parts you will be able to evaluate the integrals from parts (d) and (e).
Math 252 Lab  
Integration by Parts

From the textbook: Section 7.1, problems 57, 60, 65, and 66.
Math 252 Lab
Trigonometric Substitution, Partial Fractions, and Miscellaneous

Often integrals can be evaluated by more than one technique of integration.

1. \( \int x^3 \sqrt{x + 4} \, dx \)
   
   (a) Evaluate this integral by the substitution \( u = x + 4 \).
   
   (b) Evaluate this integral by the rationalizing substitution \( u^3 = x + 4 \).
   
   (c) Evaluate this integral by doing integration by parts twice.
       For the first integration by parts use \( u = x^2 \) \( dv = (x + 4)^{\frac{1}{3}} \, dx \).
   
   (d) Which method did you prefer for this problem? Note: This is truly a matter of opinion, so your answer may be different than my answer, and still be reasonable.

2. \( \int \frac{x}{(x^2 + 9)^{\frac{3}{2}}} \, dx \)

   (a) Evaluate this integral by the substitution \( u = x^2 + 9 \).
   
   (b) Evaluate this integral by the appropriate trig substitution.
   
   (c) Which is the better method for this integral? Note: Unlike in problem number 1, there is only one right answer to this question.

3. \( \int \frac{dx}{x^2 - 4} \)

   (a) Evaluate this integral by a partial fractions decomposition.
   
   (b) Evaluate this integral by the appropriate trig substitution.
   
   (c) Which method do you prefer for this method? Why?
4. Which method of integration should you use for $\int \frac{x}{x^2 - 9} \, dx$?
Evaluate this integral.

5. $\int \frac{dx}{(x^2 + 1)^2}$
   
   (a) Evaluate this integral by an appropriate trig substitution.

   (b) $\int \frac{dx}{(x^2 + 1)^2} = \int \frac{2x}{2x(x^2 + 1)^2} \, dx$
       
       Now evaluate this integral by first using integration by parts
       with $u = \frac{1}{2x} \quad dv = \frac{2x}{(x^2 + 1)^2} \, dx$, followed by a partial fractions.

   (c) Which method do you prefer? Why?

6. $\int \cos^2 x \, dx$
   
   (a) Use the trig identity $\cos^2 x = \frac{1}{2} + \frac{1}{2} \cos(2x)$

   (b) Use integration by parts with $u = \cos x \quad dv = \cos x \, dx$,
       then use the trig identity $\sin^2 x = 1 - \cos^2 x$,
       followed with a flip around.

   (c) Show that these answers are equivalent.

7. $\int e^{u^2} \, dx$. Let $u^2 = x$. 

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Mth 252 Lab
Techniques of Integration

Your instructor may assign a selection of problems from the practice integrals starting on the next page.

Following the problems is a list of hints. Only look after you have given a serious attempt at the integral. The solutions to these integrals are given after the hints. Don’t peek until after you are sure you have properly evaluated the integral.

Some of these integrals are quite challenging.

This is the same list of integral problems as found in Lesson 18. They are repeated here for your convenience.
The first 30 integrals do not require the use of either trigonometric substitution or partial fractions.

1. \( \int e^{5-4x} \, dx \)
2. \( \int \sin(\pi z) \, dz \)

3. \( \int \cos^2(3\theta) \, d\theta \)
4. \( \int xe^{-2x} \, dx \)

5. \( \int t^3 \ln t \, dt \)
6. \( \int \frac{4x}{\sqrt{9-4x^2}} \, dx \)

7. \( \int \frac{dx}{\sqrt{9-4x^2}} \)
8. \( \int \frac{e^t - e^{-t}}{e^t + e^{-t}} \, dt \)

9. \( \int \frac{\cos \alpha}{1 + \sin^2 \alpha} \, d\alpha \)
10. \( \int \frac{x}{1 + x^4} \, dx \)

11. \( \int x \tan^{-1} x \, dx \)
12. \( \int \sin^3 \varphi \, d\varphi \)

13. \( \int \sec^4 t \, dt \)
14. \( \int \frac{\tan(\sqrt{z})}{\sqrt{z}} \, dz \)

15. \( \int 5^{2x} \, dx \)
16. \( \int e^{3x} \sec(e^{3x}) \, dx \)
17. \[ \int \frac{1}{1 + \sqrt{x}} \, dx \]

18. \[ \int y(2 - y)^3 \, dy \]

19. \[ \int e^{3x} \sin(2x) \, dx \]

20. \[ \int \frac{t^2 + 4t + 3}{t + 1} \, dt \]

21. \[ \int x^2 e^{5x} \, dx \]

22. \[ \int \tan^2(3\theta) \, d\theta \]

23. \[ \int \frac{d\alpha}{\sec(2\alpha)} \]

24. \[ \int (x - 1)e^{-(x-1)^2} \, dx \]

25. \[ \int \tan^2 \theta \sec^2 \theta \, d\theta \]

26. \[ \int (w^2 + 2w + 1)\sqrt{w+1} \, dw \]

27. \[ \int \ln 7 \, dx \]

28. \[ \int \sec(2\theta) \, d\theta \]

29. \[ \int \ln(5x) \, dx \]

30. \[ \int \tan^5 \theta \sec^3 \theta \, d\theta \]

The next 20 integrals may require either trigonometric substitution or partial fractions.

31. \[ \int \frac{x + 3}{x^2 + 2x + 5} \, dx \]

32. \[ \int \frac{t^3}{t^2 + 1} \, dt \]

33. \[ \int \frac{1 + x}{\sqrt{1 - x}} \, dx \]

34. \[ \int \frac{dz}{z + \sqrt{z}} \]
35. \[\int \frac{d\theta}{1 + \sin \theta}\]
36. \[\int \frac{\sqrt{4-x^2}}{x^2} \, dx\]
37. \[\int \frac{x^2}{\sqrt{x^2 - 1}} \, dx\]
38. \[\int \sqrt{9-x^2} \, dx\]
39. \[\int \frac{x^3}{x^2 - 2x + 8} \, dx\]
40. \[\int \frac{x^2 + 5x}{(x-1)^3(x+1)} \, dx\]
41. \[\int \frac{dt}{(t^2 - 6t + 13)^{\frac{3}{2}}} \, dt\]
42. \[\int \frac{\cos \theta}{\sin^2 \theta + 2 \sin \theta + 2} \, d\theta\]
43. \[\int \frac{x^3 + 3x^2 - 4x + 20}{x^3 - 16} \, dx\]
44. \[\int \frac{dw}{\sqrt{w+1} - \sqrt{w}}\]
45. \[\int \frac{\sqrt{x^2 - 2x + 2}}{x-1} \, dx\]
46. \[\int \frac{\sin \theta + \tan \theta}{\cos \theta} \, d\theta\]
47. \[\int x \sin x \cos x \, dx\]
48. \[\int \frac{\sqrt{x}}{4-x} \, dx\]
49. \[\int \frac{dx}{x^3 - x^2}\]
50. \[\int \frac{dx}{(x^2 + 1)^2}\]
Hints for Practice Integrals

1. \( u = 5 - 4x \)

2. \( u = \pi z \)

3. \( \cos^2 x = \frac{1}{2} + \frac{1}{2} \cos(2x) \)

4. Parts

5. Parts

6. \( u = 9 - 4x^2 \)

7. \( \int \frac{dx}{\sqrt{1-x^2}} dx = \sin^{-1} x + C \)

8. \( u = e^t + e^{-t} \)

9. \( u = \sin \alpha \)

10. \( x^4 = (x^2)^2, \quad u = x^2 \)

11. Parts

12. \( \sin^3 \varphi = \sin^2 \varphi \sin \varphi \)

13. \( \frac{d}{dt} \tan t = \sec^2 t \)

14. \( u = \sqrt{z} \)

15. \( \int a^u du = \frac{a^u}{\ln a} + C, \quad a > 0, a \neq 1 \)

16. \( u = e^{3x} \)

17. \( u^2 = x \)

18. \( u = 2 - y \) or parts

19. Parts

20. \( t^2 + 4t + 3 = (t + 1)(t + 3) \)

21. Parts

22. \( 1 + \tan^2 x = \sec^2 x \)
23. $\sec x = \frac{1}{\cos x}$

24. $u = -(x - 1)^2$

25. $\frac{d}{dx} \tan x = \sec^2 x$

26. $x^2 + 2x + 1 = (x + 1)^2$

27. $\ln 7$ is a number

28. $u = 2\theta$

29. $\ln(5x) = \ln x + \ln 5$

30. $\frac{d}{dx} \sec x = \sec x \tan x$

31. $x^2 + 2x + 5 = (x^2 + 2x + 1) + 4$

32. Long division

33. $\sqrt{\frac{1+x}{1-x}} = \left( \frac{\sqrt{1+x}}{\sqrt{1-x}} \right) \frac{\sqrt{1+x}}{\sqrt{1-x}}$

34. $u^2 = z$

35. $\frac{1}{1 + \sin \theta} = \left( \frac{1}{1 + \sin \theta} \right) \frac{1 - \sin \theta}{1 - \sin \theta}$

36. $x = \sin \left( \frac{\theta}{2} \right)$

37. $x = \sec \theta$

38. $x = \sin \left( \frac{x}{3} \right)$

39. Long division

40. Partial fractions

41. $t^2 - 6t + 13 = (t^2 - 6t + 9) + 4$

42. $u = \sin \theta$
43. \( x^4 - 16 = \left( x^2 + 4 \right) \left( x + 2 \right) \left( x + 2 \right) \)

44. Multiply by 1

45. \( x^2 - 2x + 2 = \left( x^2 - 2x + 1 \right) + 1 \)

46. \( \tan \theta = \frac{\sin \theta}{\cos \theta} \)

47. Parts

\[
\begin{align*}
  u &= x \\
  dv &= \sin x \cos x \, dx
\end{align*}
\]

48. \( u^2 = x \)

49. \( x^3 - x^2 = x^2 (x - 1) \)

50. \( x = \tan \theta \)
The first 30 integrals do not require the use of either trigonometric substitution or partial fractions.

1. \[ \int e^{5-4x} \, dx = -\frac{1}{4} e^{5-4x} + C \]

2. \[ \int \sin(\pi z) \, dz = -\frac{1}{\pi} \cos(\pi z) + C \]

3. \[ \int \cos^2(3\theta) \, d\theta = \frac{\theta}{2} + \frac{1}{12} \sin(6\theta) + C \]

4. \[ \int xe^{-2x} \, dx = \left(-\frac{x}{2} - \frac{1}{4}\right) e^{-2x} + C \]

5. \[ \int t^3 \ln t \, dt = \frac{1}{4} t^4 \ln t - \frac{1}{8} t^4 + C \]

6. \[ \int \frac{4x}{\sqrt{9-4x^2}} \, dx = -\sqrt{9-4x^2} + C \]

7. \[ \int \frac{dx}{\sqrt{9-4x^2}} = \frac{1}{2} \sin^{-1}\left(\frac{2x}{3}\right) + C \]

8. \[ \int \frac{e^t - e^{-t}}{e^t + e^{-t}} \, dt = \ln\left(e^t + e^{-t}\right) + C \]
9. \[ \int \frac{\cos \alpha}{1 + \sin^2 \alpha} \, d\alpha = \tan^{-1} (\sin \alpha) + C \]

10. \[ \int \frac{x}{1 + x^4} \, dx = \frac{1}{2} \int \frac{du}{1 + u^2} = \frac{1}{2} \tan^{-1} u + C = \frac{1}{2} \tan^{-1} \left( \frac{x}{2} \right) + C \]

\[ u = x^2 \]
\[ du = 2xdx \]

11. \[ \int x \tan^{-1} x \, dx \]
\[ = \frac{1}{2} x^2 \tan^{-1} x - \frac{1}{2} \int \frac{x^2}{x^2 + 1} \, dx = \frac{1}{2} x^2 \tan^{-1} x - \frac{1}{2} \int 1 - \frac{1}{x^2 + 1} \, dx \]
\[ = \frac{1}{2} (x^2 + 1) \tan^{-1} x - \frac{1}{2} x + C \]

12. \[ \int \sin^3 \phi \, d\phi = \int (1 - \cos^2 \phi) \sin \phi \, d\phi = \frac{1}{3} \cos^3 \phi - \cos \phi + C \]

13. \[ \int \sec^4 t \, dt = \int (\tan^2 x + 1) \sec^2 x \, dx = \frac{1}{3} \tan^3 x + \tan(x) + C \]

14. \[ \int \frac{\tan \left( \sqrt{z} \right)}{\sqrt{z}} \, dz = 2 \ln \left| \sec \left( \sqrt{z} \right) \right| + C \]

15. \[ \int 5^{2x} \, dx = \frac{5^{2x}}{2 \ln 5} + C \]

16. \[ \int e^{3x} \sec(e^{3x}) \, dx = \frac{1}{3} \ln \left| \sec(e^{3x}) + \tan(e^{3x}) \right| + C \]
17. \[ \int \frac{1}{1 + \sqrt{x}} \, dx = 2udu = dx \quad u = \sqrt{x} \]

\[ = \int \frac{2u}{1 + u} \, du = 2 \int 1 - \frac{1}{1 + u} \, du = 2u - 2 \ln|1 + u| + C = 2\sqrt{x} - 2 \ln(1 + \sqrt{x}) + C \]

18. \[ \int (2 - y)^{\frac{3}{2}} dy = -\frac{2}{5} (2 - y)^{\frac{5}{2}} + \frac{2}{5} \int (2 - y)^{\frac{5}{2}} dy = -\frac{2}{5} (2 - y)^{\frac{5}{2}} - \frac{4}{35} (2 - y)^{\frac{7}{2}} + C \]

19. \[ \int e^{3x} \sin(2x) \, dx = \frac{3}{13} e^{3x} \sin(2x) - \frac{2}{13} e^{3x} \cos(2x) + C \]

20. \[ \int \frac{t^2 + 4t + 3}{t + 1} \, dt = \frac{1}{2} t^2 + 3t + C \]

21. \[ \int x^2 e^{5x} \, dx = \left( \frac{1}{5} x^2 - \frac{2}{25} x + \frac{2}{125} \right) e^{5x} + C \]

22. \[ \int \tan^2(3\theta) \, d\theta = \frac{1}{3} \tan(3\theta) - \theta + C \]

23. \[ \int \frac{d\alpha}{\sec(2\alpha)} = \int \cos(2\alpha) \, d\alpha = \frac{1}{2} \sin(2\alpha) + C \]

24. \[ \int (x - 1)e^{-(x-1)^2} \, dx = -\frac{1}{2} e^{-(x-1)^2} + C \]
25. \[ \int \tan^2 \theta \sec^2 \theta \, d\theta = \frac{1}{3} \tan^3 \theta + C \]

26. \[ \int (w^2 + 2w + 1) \sqrt{w + 1} \, dw = \int (w + 1)^{\frac{3}{2}} \, dw = \int (w + 1) \, dx = \frac{2}{7}(w + 1)^{\frac{7}{2}} + C \]

27. \[ \int \ln 7 \, dx = x \ln 7 + C \]

28. \[ \int \sec(2\theta) \, d\theta = \frac{1}{2} \ln \left| \sec(2\theta) + \tan(2\theta) \right| + C \]

29. \[ \int \ln(5x) \, dx = x \ln(5x) - x + C \]

30. \[ \int \tan^5 \theta \sec^3 \theta \, d\theta \]

\[ = \int \tan^4 \theta \sec^2 \theta \sec x \tan x \, dx = \int (\sec^2 \theta - 1)^{\frac{1}{2}} \sec^2 \theta \sec x \tan x \, dx \]

\[ = \int (\sec^6 x - 2 \sec^4 x + \sec^2 x) \sec x \tan x \, dx \]

\[ = \frac{1}{7} \sec^7 x - \frac{2}{5} \sec^5 x + \frac{1}{3} \sec^3 x + C \]
The next 20 integrals may require either trigonometric substitution or partial fractions.

31. \[ \int \frac{x + 3}{x^2 + 2x + 5} \, dx = \int \frac{x + 1 + 2}{(x + 1)^2 + 4} \, dx = \frac{1}{2} \ln(x^2 + 2x + 5) + \tan^{-1} \left( \frac{x + 1}{2} \right) + C \]

32. \[ \int \frac{t^3}{t^2 + 1} \, dt = \int t - \frac{t}{t^2 + 1} \, dt = \frac{1}{2} t^2 - \frac{1}{2} \ln(t^2 + 1) + C \]

33. \[ \int \sqrt{\frac{1 + x}{1 - x}} \, dx = \int \left( \frac{\sqrt{1 + x}}{\sqrt{1 - x}} \right) \left( \frac{\sqrt{1 + x}}{\sqrt{1 + x}} \right) \, dx = \int \frac{1 + x}{\sqrt{1 - x^2}} \, dx = \sin^{-1} x - \sqrt{1 - x^2} + C \]

34. \[ \int \frac{dz}{z + \sqrt{z}} = \int \frac{2u}{u^2 + u} \, du = \int \frac{2}{u + 1} \, du = 2 \ln|u + 1| + C = 2 \ln \left| 1 + \sqrt{z} \right| + C \]
   \[ u^2 = z \]
   \[ 2udu = dz \]

35. \[ \int \frac{d\theta}{1 + \sin \theta} \]
   \[ = \int \frac{1}{1 + \sin \theta} \left( \frac{1 - \sin \theta}{1 - \sin \theta} \right) \, d\theta = \int \frac{1 - \sin \theta}{1 - \sin^2 \theta} \, d\theta = \int \frac{1 - \sin \theta}{\cos^2 \theta} \, d\theta \]
   \[ = \int \sec^2 \theta - \sec \theta \tan \theta \, d\theta = \tan \theta - \sec \theta + C \]
\[ 36 \quad \int \frac{\sqrt{4-x^2}}{x^2} \, dx \quad x = 2 \sin \theta \]

\[ = \int \frac{4 \cos^2 \theta}{4 \sin^2 \theta} \, d\theta = \int \cot^2 \theta \, d\theta = \int \csc^2 \theta - 1 \, d\theta = -\cot \theta - \theta + C \]

\[ = -\frac{\sqrt{4-x^2}}{x} - \sin^{-1} \left( \frac{x}{2} \right) + C \]

\[ 37. \quad \int \frac{x^2}{\sqrt{x^2 - 1}} \, dx \quad x = \sec \theta \]

\[ = \int \sec^3 \theta \, d\theta = \frac{1}{2} \sec \theta \tan \theta + \frac{1}{2} \ln \left| \sec \theta + \tan \theta \right| + C \]

\[ = \frac{1}{2} x \sqrt{x^2 - 1} + \frac{1}{2} \ln \left| x + \sqrt{x^2 - 1} \right| + C \]

\[ 38. \quad \int \sqrt{9-x^2} \, dx \quad x = 3 \sin \theta \]

\[ = \int 9 \cos^2 \theta \, d\theta = \frac{9}{2} \theta + \frac{9}{2} \sin \theta \cos \theta + C = \frac{9}{2} \sin^{-1} \left( \frac{x}{3} \right) + \frac{1}{2} x \sqrt{9-x^2} + C \]

\[ 39. \quad \int \frac{x^3}{x^2 - 2x + 8} \, dx \]

\[ = \int x + 2 - \frac{4x + 16}{x^2 - 2x + 8} \, dx = \frac{1}{2} x^2 + 2x - 4 \int \frac{x-1 + 5}{(x-1)^2 + 7} \, dx \]

\[ = \frac{1}{2} x^2 + 2x - 2 \ln \left| (x-1)^2 + 7 \right| - \frac{20}{\sqrt{7}} \tan^{-1} \left( \frac{x-1}{\sqrt{7}} \right) + C \]
40. \[ \int \frac{x^2 + 5x}{(x-1)^2(x+1)} \, dx \]

\[ = \int \left( \frac{3}{(x-1)^2} + \frac{2}{x-1} - \frac{1}{x+1} \right) \, dx = -3 \ln |x-1| + 2 \ln |x+1| + C \]

41. \[ \int \frac{dt}{(t^2 - 6t + 13)^{3/2}} \]

\[ t = 2 \tan(t-3) \]

\[ = \int \frac{2 \sec^2 \theta}{8 \sec^3 \theta} \, d\theta = \frac{1}{4} \int \cos \theta \, d\theta = \frac{1}{4} \sin \theta + C = \frac{t-3}{4 \sqrt{t^2 - 6t + 13}} + C \]

42. \[ \int \frac{\cos \theta}{\sin^2 \theta + 2 \sin \theta + 2} \, d\theta \quad u = \sin \theta \]

\[ du = \cos \theta \, d\theta \]

\[ = \int \frac{du}{u^2 + 2u + 2} = \int \frac{du}{(u+1)^2 + 1} = \tan^{-1}(u+1) + C = \tan^{-1}(1 + \cos \theta) + C \]

43. \[ \int \frac{x^3 + 3x^2 - 4x + 20}{x^4 - 16} \, dx \]

\[ = \int \left( \frac{x-1}{x^2 + 4} - \frac{1}{x+2} + \frac{1}{x-2} \right) \, dx = \frac{1}{2} \ln(x^2 + 4) - \frac{1}{2} \tan^{-1} \left( \frac{x}{2} \right) - \ln |x+2| + \ln |x-2| + C \]
44. \[ \int \frac{dw}{\sqrt{w+1} - \sqrt{w}} \]

\[ = \int \left( \frac{1}{\sqrt{w+1} - \sqrt{w}} \right) \left( \frac{\sqrt{w+1} + \sqrt{w}}{\sqrt{w+1} + \sqrt{w}} \right) \, dw = \int \frac{\sqrt{w+1} + \sqrt{w}}{2} \, dw = \frac{2}{3} (w+1)^{\frac{3}{2}} + \frac{2}{3} w^{\frac{3}{2}} + C \]

45. \[ \int \frac{x^2 - 2x + 2}{x - 1} \, dx \quad x - 1 = \tan \theta \]

\[ = \int \frac{\sqrt{(x-1)^2 + 1}}{x - 1} \, dx = \int \frac{\sec^3 \theta}{\tan \theta} \, d\theta = \int \frac{(\tan^2 \theta + 1) \sec \theta}{\tan \theta} \, d\theta = \int \sec \theta \tan \theta + \frac{\sec \theta}{\tan \theta} \, d\theta \]

\[ = \sec \theta + \int \csc \theta \, d\theta = \sec \theta - \ln \left| \csc \theta + \cot \theta \right| + C \]

\[ = \sqrt{x^2 - 2x + 2} - \ln \left| \frac{x^2 - 2x + 2 + 1}{x - 1} \right| + C \]

46. \[ \int \frac{\sin \theta + \tan \theta}{\cos \theta} \, d\theta \]

\[ = \int \frac{\sin \theta}{\cos \theta} + \frac{\tan \theta}{\cos \theta} \, d\theta = \int \tan \theta + \sec \theta \tan \theta \, d\theta = \ln \left| \sec \theta \right| + \sec \theta + C \]

47. \[ \int x \sin x \cos x \, dx = \frac{1}{2} x \sin^2 x - \frac{1}{2} \int \sin^2 x \, dx \]

\[ = \frac{1}{2} x \sin^2 x - \frac{1}{4} x + \frac{1}{4} \sin x \cos x + C \]

48. \[ \int \frac{\sqrt{x}}{4 - x} \, dx \quad u^2 = x \]

\[ = \int \frac{2u^2}{4 - u^2} \, du = \int -2 + \frac{2}{u + 2} - \frac{2}{u - 2} \, du = -2u + 2 \ln |u + 2| - 2 \ln |u - 2| + C \]

\[ = -2\sqrt{x} + 2 \ln \left| \sqrt{x} + 2 \right| - 2 \ln \left| \sqrt{x} - 2 \right| + C \]
49. \[ \int \frac{dx}{x^3 - x^2} = \int \left( \frac{1}{x-1} - \frac{1}{x^2} - \frac{1}{x} \right) dx = \ln |x-1| - \frac{1}{x} - \ln |x| + C \]

50. \[ \int \frac{dx}{(x^2 + 1)^2} \quad x = \tan \theta \]
\[ dx = \sec^2 \theta \]
\[ = \int \frac{\sec^2 \theta}{\sec^4 \theta} \quad d\theta = \int \cos^2 \theta \quad d\theta = \frac{1}{2} \theta + \frac{1}{2} \sin \theta \cos \theta + C = \frac{1}{2} \tan^{-1} x + \frac{x}{2(x^2 + 1)} + C \]
Mth 252 Lab
Improper Integrals

From the textbook: Section 7.8, problems 57, 74, 75, and 76

For help with problems 74, 75, and 76, carefully reread Example 2 on page 572 in your textbook.
Mth 252 Lab
Exponential Growth and Decay

Remember to show all your work and to label your graphs! Just giving an answer without explanation will not receive any credit. No unlabeled graphs will be graded.

1. A sky-diver jumps out of an airplane at a height of 2000 meters with initial velocity 0 m/s (meters per second). The sky-diver’s acceleration is \( a(t) = -10e^{-0.1t} \) in the vertical direction. Here the positive direction is up.

   (a) Compute the velocity function and draw a graph of the function.

   (b) What happens to the velocity as \( t \) increases? What is the terminal velocity? (the velocity as \( t \to \infty \))? Show this on the graph and algebraically.

   (c) Explain why this is a plausible model for the velocity function of a sky-diver.

2. Newton’s Law of Cooling states that the rate at which an object cools is proportional to the difference in temperature between the object and the surrounding medium.

   When a murder is committed, the body, originally at 36° C, cools according to Newton’s Law of Cooling. Suppose that after two hours the temperature is 34° C, and that the temperature of the surrounding air is a constant 18° C.

   (a) Find the temperature, \( H \), of the body as a function of \( t \), the time in hours since the murder was committed.

   (b) Draw a graph of the temperature against time.

   (c) What happens to the temperature as the time \( t \) increases? Show this on the graph and algebraically.

   (d) Toe coroner measures the temperature of the body at 5:35 pm and finds it to be 29° C. What is her prediction for the time of the murder?
Math 252 Lab
First Order Linear Differential Equations

A first order linear differential equation is a differential equation that has the form

\[ \frac{dy}{dt} + p(t)y = g(t) \]

This type differential equations often appear in applications from many fields including engineering, physics, chemistry, economics and biology.

In this lab we will walk through a process that can be used to find the solution to these differential equations. The reason that this process works is explained in the appendix to this lab.

Step 1: Find an antiderivative to \( p(t) \). Call this antiderivative \( a(t) \).

Step 2: Let \( \mu(t) = e^{a(t)} \). If needed simplify this expression.

Step 3: Evaluate the definite integral \( \int \mu(t)g(t) \, dt \). Be sure to include the constant of integration.

Step 4: Divide the result from step 3 by \( \mu(t) \) and set this equal to \( y \). This is the solution to the differential equation.

Step 5: Check your answer.

Here's an example:

\[ \frac{dy}{dt} + \frac{3}{t} y = t^2, \quad t > 0 \]

Step 1: \( \int \frac{3}{t} \, dt = 3\ln t + K \). Take \( a(t) = 3\ln t \)

Step 2: \( \mu(t) = e^{3\ln t} = e^{\ln t^3} = t^3 \)

Step 3: \( \int(t^3)(t^2) \, dt = \int t^5 \, dt = \frac{t^6}{6} + C \)

Step 4: \( y = \frac{1}{t^3} \left( \frac{t^6}{6} + C \right) \) Then \( y = \frac{t^3}{6} + Ct^{-3} \)

Step 5: Check
\[
\frac{dy}{dt} + \frac{3}{t} y = \frac{d}{dt} \left( \frac{t^3}{6} + Ct^{-3} \right) + \frac{3}{t} \left( \frac{t^3}{6} + Ct^{-3} \right)
\]
\[
= \frac{t^2}{2} - 3Ct^{-4} + \frac{t^2}{2} + Ct^{-4}
\]
\[
= t^2 \quad \text{as needed}
\]
1. Solve the differential equations: \[ \frac{dy}{dt} - \frac{3}{t}y = \ln t, \quad t > 0 \]

Step 1:

Step 2:

Step 3:

Step 3:

Step 5:
2. Solve the differential equations: \( \frac{dy}{dx} + x y = x \)

Step 1:

Step 2:

Step 3:

Step 3:

Step 5:
3. Solve the differential equations: \( \frac{dy}{d\theta} + \tan(\theta) \cdot y = \cos^2(\theta), \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2} \)

Step 1:

Step 2:

Step 3:

Step 3:

Step 5:
Appendix to lab on First Order Linear Differential equations:

In the following discussion please keep in mind that $y$, $p$, $g$ and $\mu$ are all functions of $t$.

Consider the first order linear differential equation, $y' + py = g$. The secret is to make the observation that the left hand side of this equation almost looks like the result of a product rule differentiation.

What we will do is multiply the equation by a currently unknown function, $\mu$ with the idea that we will determine which function $\mu$ will give us a product rule. The function, $\mu$, is called an integrating factor.

Our differential equation now has the form:

$$\mu y' + \mu py = \mu g$$

By the product rule: $$\frac{d}{dt}(\mu y) = \mu y' + \mu' y$$

So we want $\mu y' + \mu' y = \mu y' + \mu py$

In order for this equation to be satisfied we must have that $\mu' = \mu p$.

This is a differential equation for $\mu$ we can solve:

$$\mu' = \mu p$$

$$\frac{\mu'}{\mu} = p$$

$$\int \frac{\mu'(t)}{\mu(t)} dt = \int p(t) dt$$

$$\ln|\mu(t)| = \int p(t) dt + K$$

$$|\mu(t)| = e^{\int p(t) dt + K} = e^K e^{\int p(t) dt} = C e^{\int p(t) dt}$$

By taking $C$ to be plus or minus 1 as needed we see that the needed integrating factor is $\mu(t) = e^{\int p(t) dt}$.

Now we have:

$$\frac{d}{dt}(y e^{\int p(t) dt}) = g e^{\int p(t) dt}$$
Integrating gives:

\[ ye^{\int p(t)\,dt} = \int ye^{\int p(t)\,dt} \, dt + C \]

Then:

\[ ye^{\int p(t)\,dt} = \int ge^{\int p(t)\,dt} \, dt + C \]

\[ y = \frac{\int ge^{\int p(t)\,dt} \, dt}{e^{\int p(t)\,dt}} + \frac{C}{e^{\int p(t)\,dt}} \]

Which is usually written as:

\[ y = e^{-\int p(t)\,dt} \int ge^{\int p(t)\,dt} \, dt + Ce^{-\int p(t)\,dt} \]