Contingent Claims On Assets With Conversion Costs

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Abstract

Contingent claim contracts on certain natural resources may involve underlying assets which exist in either *developed* (S) or *undeveloped* (S-c) states. While the positive prices of the developed asset may be modelled by the standard geometric Brownian motion over the specified term of the contract, the lower priced undeveloped asset may become negative due to a conversion cost (c) given in the contracts. In particular pricing such contracts involves natural arbitrage opportunities on undeveloped assets in the event that developed prices drop below c and then rise before expiry T. Thus we consider the effect on contract price of requiring contract settlement (with possible rebates) at time $\tau_c = \inf\{t \leq T : S(t) = c\}$. In particular, we obtain the final-value pricing equation (with knock-out boundary)

$$\frac{\partial \pi}{\partial t} + rs\frac{\partial \pi}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 \pi}{\partial s^2} - r\pi = rc\frac{\partial \pi}{\partial s}$$

where r is the risk free interest rate, σ the volatility in developed asset. Mathematically, the success of this approach is based on martingale localization of the classic theory of Harrision-Pliska (1981) to provide a simple and precise meaning of (localized) arbitrage-free pricing in this context.

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1 Introduction

The valuation of a large number of diverse contingent claim contracts on underlying assets with strictly positive prices rests on the general mathematical foundation given in the classic paper of Harrison and Pliska (1981). However the emergence of contracts in applications to the so called real options has lead to an interest in adapting this theory to contracts on assets with non-positive prices, e.g. electric power options, contracts on natural resources.

The particular application which serves to motivate this paper is that of pricing federal timber leases. Such contracts are written on an asset which has a positive *developed* value S and an *undeveloped* value $S-c$, where $c \geq 0$ represents a one-time fixed conversion cost. This cost c is specified in the contract and is quite distinct from the recurring transaction costs familiar in financial markets.

Morck, Schwartz and Stangeland (1989) obtained a valuation of such contracts using developed assets as a hedging instrument. However this approach also required use of an adhoc modelling parameter (convenience yield) to obtain the price; see Morck et al (p 478, 1989) and Banks (p 236 - 240, 2000).

In the absence of convenience yields, the portfolio match on delivery of undeveloped assets is achieved with a price that is insufficient for hedging with developed assets. Noticing that this effect is directly traceable to conversion costs, Burnes, Thomann, and Waymire(1999) were led to consider the following pricing model based on hedges with the undeveloped asset:

$$\frac{\partial \pi}{\partial t} + rs\frac{\partial \pi}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 \pi}{\partial s^2} - r\pi = rc\frac{\partial \pi}{\partial s}, \tag{1.1}$$

where r is the risk free interest rate, σ the volatility in the developed asset, and c the conversion cost. In particular by explicit considerations of the conversion cost c specified in the contracts, this price avoids the use of the convenience yield parameter.

As previously noted, undeveloped assets such as timber illustrate assets which may be stored and which may take non-positive values. Since storability of valueless assets generates obvious arbitrage opportunities over an arbitrary but fixed time period, one must further restrict the contracts. Specifically, our model imposes a contractual specification of a knock-out boundary at $S = c$, together with a possible rebate at settlement. With this simple idea we may show that these contracts are "arbitrage free" up to a natural random settlement time via martingale localization of the Harrison-Pliska (1981) theory.

So, in the present paper we provide a simple and mathematically complete derivation of the pricing equation and identify the precise meaning of an arbitrage-free solution. As a result of this mathematical analysis we also obtain an interesting new equivalent pricing equation which relates (1.1) to the standard Black-Scholes-Merton equation with drift parameter r but a non-constant volatility $\sigma^2(s)$. We will also see that although the contracts are intrinsically exercised as European options, (1.1) may be viewed mathematically in terms of a transformation to Asian type contracts. Of course when the conversion cost $c = 0$ this framework trivially reduces to that of the standard Black-Scholes-Merton equation. A third equation, obtained by pricing under the standard change of measure to make the discounted developed asset a martingale, has advantages for numerical implementation which will also be discussed.

The organization of the paper is as follows. Section 2 gives a precise mathematical formulation of the valuation problem and introduces notation. The pricing equation (1.1) with a Dirichlet boundary condition is derived in Section 3. In particular by localization one may obtain an equivalent local martingale measure for the discounted undeveloped asset and the discounted portfolio and, hence, a locally arbitrage free discounted expected value solution. This valuation is also shown to mathematically lead to consideration of Asian type options with a down and out barrier. Section 4 contains a binomial tree numerical approximation for the valuation of these contracts and discusses its merits and limitations.

2 Mathematical Formulation of the Problem

We suppose that the developed asset evolves according to a geometric Brownian motion $\{S(t) : t \geq 0\}$ defined on a probability space $(\Omega, \mathcal{F}, P)$. That is,

$$S(t) = S_0 \exp\{(\mu - \frac{\sigma^2}{2})t + \sigma W(t)\},$$

where $\mu > 0$ is the *mean yield rate,* $\sigma > 0$ the *volatility,* and $\{W(t) : t \geq 0\}$ is a standard Brownian motion. Equivalently

$$dS = \mu S\, dt + \sigma S dW, \qquad S(0) = S_0. \tag{2.2}$$

The undeveloped asset value is defined by $S - c$, where $c \geq 0$ represents the conversion cost.

We consider European contracts on a possibly non-positive (undeveloped) asset with expiry $T > 0$ whose intrinsic value V at expiry is prescribed by the contract. In the case of the forest lease considered by Burnes etal (1999), for example, V has the form $V = -V_1 + \frac{1}{2}V_2$, where V_1 is payoff on a put, and V_2 a call payoff. Since the undeveloped asset value $S - c$ may take a non-positive value prior to expiry T, the contract will be declared settled if $S(t) \leq c$ for some $t \leq T$, i.e. a down and out knock-out barrier is imposed at c and settlement of all accounts will be assumed, including a possible rebate amount $g(\tau_c)$ at such time τ_c, where

$$\tau_c = \inf\{t \geq 0 : S(t) = c.\}$$

This localizes the no-arbitrage considerations to the random settlement time $\tau_c \wedge T$. By localizing the arbitrage period, one may define no-arbitrage valuations for the period of time until contract settlement. This will be made precise through the use of local martingale theory.

Let $B(t)$ denote the time t value of a risk free security which evolves at a given interest rate $r > 0$. That is,

$$B(t) = e^{rt}B_0, \qquad t \geq 0.$$

Let $\mathcal{F}_t = \sigma\{W(s) : s \leq t\}$. A *localized trading strategy* is a triple (φ, ψ, τ_c), where τ_c is a stopping time and $\{(\varphi(t), \psi(t)) : 0 \leq t \leq \tau_c \wedge T\}$ is $\mathcal{F}_t$-adapted a right continuous (bivariate) process such that $\int_0^{T\wedge\tau_c}(\varphi^2(t) + \psi^2(t))dt < \infty$; the amount of asset φ is referred to as the portfolio *delta*. In constructing the hedging portfolio comprised of ψ risk free bonds and φ units of undeveloped asset, the localized arbitrage considerations require that the value of the portfolio given by

$$\pi(t, S(t)) = \varphi(t)(S(t) - c) + \psi(t)B(t), \qquad t \leq \tau_c \wedge T, \tag{2.3}$$

evolve to the prescribed payoff by time $\tau_c \wedge T$,

$$\begin{aligned} \pi(T \wedge \tau_c, S(\tau_c \wedge T)) &= \tilde{V}(T \wedge \tau_c, S(T \wedge \tau_c)) \\ &:= V(S(T))\mathbf{1}[\tau_c > T] + g(\tau_c)\mathbf{1}[\tau_c \leq T], \end{aligned} \tag{2.4}$$

subject to the self-financing condition defined as

$$d\pi = \varphi(t)dS + \psi(t)dB(t), t < \tau_c \wedge T. \tag{2.5}$$

We shall show that in fact the function $\pi(t,s),\ t \geq 0,\ s \geq c$ solves (1.1) with final value

(2.6) $$\pi(T,s) = V(s),\ t \geq 0.$$

and knock-out and rebate boundary value with rebate at c

(2.7) $$\pi(t,c) = g(t),\ t \geq 0.$$

Moreover, we will show that the hedging portfolio based on undeveloped assets and risk free bonds is obtained by

$$\varphi(t) = \frac{\partial \pi}{\partial s}(t, S(t)), \quad \psi(t) = e^{-rt}\left(\pi(t, S_t^{(c)}) - \varphi S_t^{(c)}\right), 0 \leq t \leq \tau_c \wedge T.$$

Remark: Note that with an appropriate rebate structure, forwards can be used to hedge contracts on undeveloped assets in a similar fashion as with forwards on stocks. Indeed, consider a payoff

(2.8) $$s - c - K$$

with a rebate on $s = c$

(2.9) $$g(t) = Ke^{-r(T-t)}$$

A simple calculation shows that

$$\pi(t,s) = s - c - Ke^{-r(T-t)}$$

satisfies the equation (1.1) with final data (2.8) and boundary condition (2.9). Requiring that the strike price is such that the initial value of a forward vanishes leads immediately to the following application of (1.1),(2.6), (2.7).

Proposition 2.1 *Assume that $\pi(t,s)$ is a solution of (1.1) on $0 < t < T, s > c$ with final data $\pi(T,s) = s - c - K$ and boundary condition $\pi(t,c) = e^{-r(T-t)}K$. Then, for a fixed $S_0 > c$, the strike price of a forward such that $\pi(0, S_0) = 0$ is $K = (S_0 - c)e^{rT}$.*

It is simple to show that a replicating portfolio can be obtained using forwards with the rebate structure given by (2.9).

One last bit of notation is to let Q denote the equivalent martingale measure for the discounted developed asset $\{e^{-rt}S(t) : 0 \leq t \leq T\}$. Under Q one has

(2.10) $$dS = rSdt + \sigma SdW, \qquad t \geq 0.$$

Although in the presence of the cost c one may not expect Q to serve as an equivalent martingale measure for the localized discounted portfolio, this is a natural choice to begin the analysis. In the next section we will obtain the appropriate change of measure as a change of measure from Q to an equivalent local martingale measure $Q^{(c)}$ for the localized discounted undeveloped asset $S - c$ and corresponding hedging portfolio.

3 Existence Of Equivalent Local Martingale Measure.

As remarked at the end of the last section, Q will denote the usual change of measure for which the developed asset obeys (2.10). In particular, Q is the equivalent martingale measure which makes the discounted price of the developed asset $\{e^{-rt}S(t) : 0 \leq t \leq T\}$ a martingale. In order to obtain the price which is arbitrage free up to settlement time based on the undeveloped asset and a prescribed rebate at settlement time, we shall produce an equivalent probability $Q^{(c)}$ under which (i.) the discounted process $\{e^{-rt}(S(t) - c) : 0 \leq t \leq T\}$ and (ii.) the discounted portfolio $\{e^{-rt}\pi(t, S(t)) : 0 \leq t \leq T\}$ are local martingales on $[0, \tau_c \wedge T]$. For this it is convenient to define the positive stopped process

$$Z(t) = S(t) - c, 0 \leq t < T \wedge \tau_c. \tag{3.1}$$

Proposition 3.1 *Let Q be the probability measure on $\mathcal{F}_T$ such that $\{Y(t) = e^{-rt}S(t) : 0 \leq t \leq T\}$ is a martingale. Then there is an equivalent probability measure $Q^{(c)}$ such that $e^{-rt}Z(t) : 0 \leq t < T \wedge \tau_c\}$ is a positive (local) martingale.*

Proof. Under Q one has on $[0, T]$ that

$$d(e^{-rt}S(t)) = \sigma e^{-rt}S(t)dW(t).$$

Since τ_c is a stopping time it is a well-known consequence of Doob's Optional Stopping theorem that $\{Y^{(c)}(t) := e^{-r(t\wedge\tau_c)}S(t \wedge \tau_c) : 0 \leq t \leq T\}$ is a martingale with respect to the same filtration $\mathcal{F}_t, 0 \leq t \leq T$. Thus, on $[0, T \wedge \tau_c]$ one has

$$d(e^{-rt}Z(t)) = \sigma e^{-rt}S(t)dW(t) + rce^{-rt}dt.$$

In particular, therefore,

$$d(e^{-rt}Z(t)) = \sigma e^{-rt}(Z(t)+c)dW(t) + rce^{-rt}dt.$$

The assertion now follows by an application of the Cameron-Martin-Girsanov theorem. ■

It is interesting to observe that under $Q^{(c)}$ one obtains the volatility coefficient for the price of the localized undeveloped asset $Z = S - c$ as

$$\sigma(z) = \frac{z+c}{z}\sigma. \tag{3.2}$$

In particular $\sigma(z) = \sigma$ if and only if $c = 0$.

Theorem 3.1 *Let Q be the probability measure on $\mathcal{F}_T$ such that $\{Y(t) = e^{-rt}S(t) : 0 \le t \le T\}$ is a martingale, and let $Q^{(c)}$ be the equivalent probability obtained in the previous proposition such that $e^{-r(t\wedge\tau_c)}(S(t\wedge\tau_c)-c)$ is a (local) martingale. Then,*

$$\pi(t, S(t)) = e^{rt}\mathbf{E}_{Q^{(c)}}\left\{\mathbf{1}[\tau_c < T]e^{-r\tau_c}g(\tau_c) + \mathbf{1}[\tau_c \ge T]e^{-rT}V\left(S(T)\right)|S(t)\right\}$$

satisfies the differential equation (1.1) with initial data (2.6) and boundary condition (2.7) and $M(t\wedge\tau_c) := e^{-r(t\wedge\tau_c)}\pi\left(t\wedge\tau_c, S(t\wedge\tau_c)\right)$, $0 \le t \le T\wedge\tau_c$, is also a $Q^{(c)}$ (local) martingale.

Proof: First let us consider the diffusion $\{Z(t) : 0 \le t \le \tau_c \wedge T\}$ given under $Q^{(c)}$ by

$$dZ = rZdt + \sigma(Z+c)dW, \quad Z(0) = s - c \ge 0. \tag{3.3}$$

Then standard semigroup theory and Ito calculus yield the following; eg. see Freidlin (1985), Friedman (1975). Let

$$\begin{aligned}\tilde{\pi}(t,z) \quad &= \mathbf{E}\left\{\mathbf{1}[\tau_c < T]e^{-r(\tau_c - t)}g(\tau_c)|Z(t) = z\right\}\\ &+\mathbf{E}\left\{\mathbf{1}[\tau_c \ge T]e^{-r(T-t)}V\left(Z(T)+c\right)|Z(t) = z\right\}.\end{aligned} \tag{3.4}$$

Then $\tilde{\pi}(t,z),\ t \ge 0,\ z \ge 0$ solves

$$\frac{\partial\tilde{\pi}}{\partial t} + rz\frac{\partial\tilde{\pi}}{\partial z} + \frac{1}{2}\sigma^2(z+c)^2\frac{\partial^2\tilde{\pi}}{\partial z^2} - r\tilde{\pi} = 0, \tag{3.5}$$

with final value

$$\tilde{\pi}(T,z) = V(z+c),\ z \ge 0. \tag{3.6}$$

and knock-out and rebate boundary value with rebate at $z = 0$, i.e. $s = c$,

$$\tilde{\pi}(t, 0) = g(t), \ t \geq 0. \tag{3.7}$$

With this one may define

$$\pi(t, s) = \tilde{\pi}(t, z + c),$$

and simply note that the differential equation (1.1) with initial data (2.6) and rebate (2.7) follows from (3.5) under the change of variable $s = z + c$. Finally, an application of the Doob's Optional Stopping theorem shows that M is a local martingale as well. ∎

Remark The above derivation of (1.1) both simplifies and completes the treatment given in Burnes etal (1999).

In view of the Harrison, Pliska (1981) martingale theory we obtain from the positive (local) martingale $e^{-rt}(S(t) - c), 0 \leq t < T \wedge \tau_c$, that the pricing is arbitrage free up to the settlement time as follows.

Corollary 3.1 *The market model is arbitrage free on the interval* $[0, T \wedge \tau_c)$ *under the pricing formula (3.4)*

Proof: To see that the price is arbitrage free on $[0, \tau_c \wedge T]$ observe from the previous theorem that if there is a localized strategy (φ, ψ) such that, cf (2.4),

$$\tilde{V}_0 = \varphi_0(S_0 - c) + \psi_0 B_0 = 0, S_0 = s > c,$$

then by the martingale property one has

$$\mathbf{E} e^{-r(\tau_c \wedge T)} \tilde{V}_{\tau_c \wedge T} = 0.$$

This is the definition of arbitrage free up to settlement time. ∎

To complete the pricing we identify the delta in the replicating portfolio under localization with the following proposition. This also yields an alternative pricing formula under Q whose relevance will be discussed in the context of numerical computation in the next section.

Proposition 3.2 *Let*

$$M(t) = e^{-rt}\pi(t, S(t)) - rc\int_0^t \varphi(u) e^{-ru} du, 0 \leq t \leq T \wedge \tau_c.$$

Then under the probability measure Q defined by (2.10), one has that the process $\{M(t \wedge \tau_c) : 0 \leq t \leq T\}$ is a uniformly integrable martingale with respect to $\mathcal{F}_t, t \geq 0$, and

$$\varphi = \frac{\partial \pi}{\partial s}(t, S_t^{(c)}), \text{ and } \psi = e^{-rt}\left(\pi(t, S_t^{(c)}) - \varphi S_t^{(c)}\right),$$

Proof. In view of self-financing (2.5) and (2.4), (2.10), one has that $EM(t) < \infty, t \leq T$, and

$$\begin{aligned} dM &= -re^{-rt}\pi dt + e^{-rt}d\pi - rc\varphi(t)e^{-rt}dt \\ &= -re^{-rt}\pi dt + e^{-rt}(\varphi dS + \psi dB) - rc\varphi e^{-rt}dt \\ &= -re^{-rt}\varphi(S - c)dt - re^{-rt}\psi B dt \\ &\quad + e^{-rt}\varphi dS + e^{-rt}\psi dB - rc\varphi e^{-rt}dt \\ &= -re^{-rt}\varphi S dt + e^{-rt}\varphi(rSdt + \sigma S dW) \\ &= e^{-rt}\varphi(t)\sigma S dW. \end{aligned} \tag{3.8}$$

Now define a martingale $\{Y(t) : 0 \leq t \leq T\}$ by

$$Y(t) = M(0) + \int_0^t e^{-rs}\varphi(s \wedge \tau_c)\sigma S(s) dW(s), \quad 0 \leq t \leq T.$$

Then in view of (3.8) one has that $Y(t) = M(t)$ for $0 \leq t < \tau_c \wedge T$, and by continuity this equality extends to $t = \tau_c \wedge T$ as well. That is

$$M(t \wedge \tau_c) = Y(t \wedge \tau_c), 0 \leq t \leq T.$$

Now apply Doob's optional stopping, in particular see Stroock and Varadhan(Cor1.2.7, p. 26, 1979), to see that $\{Y(t \wedge \tau_c) : 0 \leq t \leq T\}$ is a martingale and, consequently, $\{M(t \wedge \tau_c) : 0 \leq t \leq T\}$ is a martingale, i.e. $\{M(t) : 0 \leq t \leq \tau_c \wedge T\}$ is a local martingale.

Using (3.8) one has

$$M(t \wedge \tau_c) = M(0) + \int_0^t e^{-ru}\varphi(u)\sigma S(u) dW(u). \tag{3.9}$$

Note that by Ito's lemma applied to the function $e^{-rt}\pi(t, S(t))$ on $[\tau_c \leq t]$ one has, in integral form,

$$e^{-rt}\pi(t, S(t)) = \pi(0, S_o) + \int_0^t e^{-ru}\left\{\frac{\partial \pi}{\partial t}(u, S(u)) - r\pi(u, S(u))\right.$$

$$+rS(u)\frac{\partial\pi}{\partial s}(u,S(u))+\frac{1}{2}\sigma^2S^2(u)\ \frac{\partial^2\pi}{\partial s^2}(u,S(u))\Bigg\}\,du$$

$$+\int_0^t e^{-ru}\sigma S(u)\frac{\partial\pi}{\partial s}(u,S(u))dW. \tag{3.10}$$

Thus, on $[\tau_c > t]$ one has using the definition of $M(t)$ and (3.10),

$$M(t)=\pi(0,S_0)+\int_0^t a(u)du+\int_0^t e^{-ru}\sigma S^{(c)}(u)\frac{\partial\pi}{\partial s}(u,S^{(c)}(u))dW, \tag{3.11}$$

where

$$\begin{aligned} a(t) \;=\; & \int_0^t e^{-ru}[\frac{\partial\pi}{\partial t}\left(u,S^{(c)}(u)\right)-r\pi\left(u,S^{(c)}(u)\right) \\ & +rS^{(c)}(u)\frac{\partial\pi}{\partial s}\left(u,S^{(c)}(u)\right)+\frac{1}{2}\sigma^2S^{(c)2}(u)\frac{\partial^2\pi}{\partial s^2}\left(u,S^{(c)}(u)\right)]du \\ & -rc\int_0^t \varphi(u)e^{-ru}du. \end{aligned}$$

In particular

$$M(t\wedge\tau_c)=\pi(0,S_0)+\int_0^{t\wedge\tau_c} a(u)du+\int_0^{t\wedge\tau_c} e^{-ru}\sigma S^{(c)}(u)\frac{\partial\pi}{\partial s}(u,S^{(c)}(u))dW.$$

Thus, by uniqueness in the Martingale Representation Theorem applied to $\{M(t\wedge\tau_c) : 0\le t\le T\}$, eg. see Karatzas and Shreve (1991), or Rogers and Williams (1987), we have

$$e^{-ru}\sigma S^{(c)}(u)\frac{\partial\pi}{\partial s}(u,S^{(c)}(u))=e^{-ru}\varphi(u)\sigma S^{(c)}(u)$$

and therefore on $[\tau_c\ge u]$

$$\varphi(u)=\frac{\partial\pi}{\partial s}(u,S^{(c)}(u)). \quad \blacksquare$$

We conclude this section with a few technical remarks on the form of the pricing formula.

Remark: Rewriting (1.1) in the form

$$\frac{\partial\pi}{\partial t}+r(s-c)\frac{\partial\pi}{\partial s}+\frac{1}{2}\sigma^2s^2\frac{\partial^2\pi}{\partial s^2}-r\pi=0, \tag{3.12}$$

suggests consideration of another natural diffusion $\{X(t) : t \geq 0\}$ defined under $Q^{(c)}$ by

(3.13) $$dX = r(X - c)dt + \sigma X dW$$

By a transformation between Stratonovich and Ito calculus, eg. see Ikeda and Watanabe (1981), one may write

$$\begin{aligned} X(t) &= X(0)\exp\left\{\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W(t)\right\} \\ &\quad - rc\int_0^t \exp\left\{\sigma(W(t) - W(u)) + \left(r - \frac{1}{2}\sigma^2\right)(t-u)\right\} du \\ &= X(0)L^{(t)}(0) - rc\int_0^t L^{(t)}(u)du \end{aligned}$$

where

$$L^{(t)}(u) = \exp\{\sigma(W(t) - W(u)) + (r - \frac{1}{2}\sigma^2)(t-u)\}.$$

Note that $\left\{\int_0^t L^{(t)}(u)du : t \geq 0\right\}$ is distributed as the averaged process given by $\left\{A^{(\nu)}(t) = \int_0^t \exp\left(W(u) + \nu u\right) du : t \geq 0\right\}$, where $\nu = r - \frac{1}{2}\sigma^2$. Moreover, $L^{(t)}(0)$ is *independent* of $L^{(t)}(u)$. With this the value of the portfolio from Proposition 3.1 may be expressed with expectations under $Q^{(c)}$ as

$$\begin{aligned} \pi(t,s) &= \mathbf{E}\left\{\mathbf{1}[\rho_c < T]e^{-r(\rho_c - t)}g(\rho_c)|X(t) = s\right\} \\ &\quad + \mathbf{E}\left\{\mathbf{1}[\rho_c \geq T]e^{-r(T-t)}V\left(sL^{(T)}(t) - rc\int_t^T L^{(T)}(u)du\right)|X(t) = s\right\}, \end{aligned} \quad (3.14)$$

where $\rho_c = \inf\{t : X(t) = c\}$. Thus, expressed in this way one observes the valuation appears to have an inherently Asian structure. While we do not have explicit formulae for the discounted expected values of functions of the stopped process $\left\{X(t) : 0 \leq t < T \wedge \rho^{(c)}\right\}$, recent approaches of Yor (1992), Geman and Yor (1992,1993), Rogers and Shi (1995), Gould (1999), Bhattacharya, Thomann, and Waymire (2001) calculate these as well as transition probabilities for Asians $\{X(t) : t \geq 0\}$ without barriers, which in turn give an explicit upper bound on the price (3.14). It would be useful to extend these calculations to the stopped process. However at this stage we will consider numerical discretizations in the next section.

Remark: Note that by Ito's lemma

$$X(t) = S(t)(1 - rc\int_0^t \frac{du}{S(u)})$$

solves (3.3) so that ρ_c can be defined in terms of $S(t)$ as

$$\rho_c \equiv \rho_c(S) = \inf\{t \le T : X(t) \equiv S(t)(1 - rc\int_0^t \frac{du}{S(u)}) = c\}.$$

In fact, apart from the starting condition, this is the Shiryaev-Roberts equation cited in Karatzas and Shreve (p.168, 1991). One may also easily check that

$$S(t) = X(t)\exp\{rc\int_0^t \frac{du}{X(u)}\}, \quad t \le \rho_c.$$

4 Discrete Approximation

The numerical implementation of the equation (1.1) is itself an interesting problem. One has a number of approaches available such as Monte Carlo estimation of the discounted expected values, possibly quasi-Monte Carlo approaches, and various discretizations schemes, eg. see Broadie, Glasserman, Kou (1999). The purpose of this section is to introduce discrete time approximations to the valuation formulas given in the previous section and discuss the practicality of the approximations for numerical computations.

We use the standard Cox-Ross-Rubinstein (1979) tree approximation for the asset values. The interval $[0,T]$ is considered to be made out of N trading intervals $[j\Delta t,\ (j+1)\Delta t],\ j = 0,\dots,N-1, T = N\Delta t$. We denote by $S_j,\ j = 0,1,...,N$ the value of the developed asset at time $j\Delta t$. In the CRR model, the value at time $(j+1)\Delta t$ of the developed asset would be at two possible states, $S_j u$ or $S_j d$ corresponding to an upward or downward movement. We also denote by $R = e^{r\Delta t}$ the interest accrued on an interval of length Δt, and assume that $d < 1 < R < u$.

Let's first note the need for the rebate boundary condition at $S = c$ to prevent arbitrage opportunities. Consider a portfolio holding φ_j units of the undeveloped asset and ψ_j units of a riskless bond on the trading interval $[j\Delta t,\ (j+1)\Delta t]$. Its worth is

$$\pi(j, S_j) = \varphi_j(S_j - c) + \psi_j B_j$$

If the value of the option is not set the first time $S_j \le c$, simple arbitrage opportunities would exist. Simply replacing the process $\{S_j\}$ by $\{(S_j - c)^+\}$ would not prevent these arbitrage opportunities either. Indeed, if this were the case and if for some $j, S_j \le c$, φ_j can be set to any arbitrary value without changing the value of the portfolio. However, with a positive probability, $S_k > c$ for some $N > k > j$ and thus there is a riskless profit with positive probability.

We therefore consider the discrete process absorbed at $S = c$ denoted, as in previous sections, by $S_j^{(c)}$. We also assume that given parameters u, d, S_0, T and c, the selection of Δt is such that

$$S_0 d^k = c \text{ for some } k \geq 0.$$

As in previous sections let $\tau_c = \min\{k \geq 0 : S_k = c\}$. The construction of the self-financing portfolio that matches the payoff

$$\tilde{V}(N \wedge \tau_c, S_{N \wedge \tau_c}) := V(S_N)\mathbf{1}[\tau_c > k] + g(\tau_c)\mathbf{1}[\tau_c \leq k]$$

follows standard methods. In particular, the self-financing condition can be written as

$$\Delta\pi = \varphi_j(S_{j+1} - S_j) + \psi_j B_j (R - 1) \tag{4.1}$$

where

$$\varphi_j = \left.\frac{\Delta\pi}{\Delta S}\right|_{S_j} = \frac{\pi(j+1, S_j u) - \pi(j+1, S_j d)}{S_j u - S_j d} \tag{4.2}$$

We can now state the analogue of Proposition 3.2. We denote by $Q^{(0)}$ the probability under which the process $R^{-j}S_j$ is a martingale.

Proposition 4.1 *Assume that the self-financing condition (4.1) holds and on $\tau_c \geq k, k \leq N$, let*

$$M(k) = R^{-k}\pi\left(k, S_k\right) - \frac{(R-1)}{R} c \sum_{j=0}^{k-1} \left.\frac{\Delta\pi}{\Delta S}\right|_{S_j} R^{-j}.$$

Then $M(k \wedge \tau_c)$ for $k \leq N$ is a $Q^{(0)}$ martingale.

Proof. Since

$$\begin{aligned}
\mathbf{E}_{Q^{(0)}}\left\{M(k \wedge \tau_c)|\mathcal{F}_{k-1}\right\} &= \mathbf{E}_{Q^{(0)}}\left\{M(\tau_c)\mathbf{1}[\tau_c < k]|\mathcal{F}_{k-1}\right\} \\
&\quad + \mathbf{E}_{Q^{(0)}}\left\{M(k)\mathbf{1}[\tau_c \geq k]|\mathcal{F}_{k-1}\right\} \\
&= M(\tau_c)\mathbf{1}[\tau_c < k] \\
&\quad + \mathbf{1}[\tau_c \geq k]\mathbf{E}_{Q^{(0)}}\left\{M(k)|\mathcal{F}_{k-1}\right\}
\end{aligned}$$

it is enough to show that on $\tau_c \geq k, k \leq N$

$$\mathbf{E}_{Q^{(0)}} \left(M(k) | \mathcal{F}_{k-1} \right) = M(k-1)$$

Indeed,

$$\mathbf{E}_{Q^{(0)}} \left\{ M(k) | \mathcal{F}_{k-1} \right\} = R^{-k} \mathbf{E}_{Q^{(0)}} \left\{ \pi \left(k, S_k \right) | S_{k-1} \right\} - \frac{R-1}{R} c \sum_{j=0}^{k-1} \left. \frac{\Delta \pi}{\Delta S} \right|_{S_j} R^{-j}$$

Using (4.1) we have

$$\begin{aligned}
\mathbf{E}_{Q^{(0)}} \left\{ \pi \left(k, S_k \right) | S_{k-1} \right\} &= \mathbf{E}_{Q^{(0)}} \left\{ \varphi_{k-1} \left(S_k - c \right) | S_{k-1} \right\} + R \psi_{k-1} B_k \\
&= \varphi_{k-1} \left(R S_{k-1} - c \right) + R \psi_{k-1} B_k \\
&= R \left(\varphi_{k-1} \left(S_{k-1} - c \right) + \psi_{k-1} B_{k-1} \right) \\
&\quad + (R-1) c \varphi_{k-1} \\
&= R \pi (k-1, S_{k-1} + (R-1) c \left. \frac{\Delta \pi}{\Delta S} \right|_{S_{k-1}}
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathbf{E}_{Q^{(0)}} \left\{ M(k) | \mathcal{F}_{k-1} \right\} &= R^{-(k-1)} \pi (k-1, \ S_{k-1}) \\
&\quad - \frac{R-1}{R} c \sum_{j=0}^{k-2} \left. \frac{\Delta \pi}{\Delta S} \right|_{S_j} R^{-j} = M(k-1)
\end{aligned}$$

as claimed. ∎

Corollary 4.1 *Under the assumptions of Proposition 4.1, and with* $s > c$, $k \leq N$

$$\begin{aligned}
\pi(k, s) &= \mathbf{E}_{Q^{(0)}} \left\{ R^{-(N \wedge \tau_c - k \wedge \tau_c)} \tilde{V} \left(N \wedge \tau_c, S_{N \wedge \tau_c} \right) | S_k = s \right\} \\
&- \frac{R-1}{R} c \mathbf{E}_{Q^{(0)}} \left\{ \sum_{j=k}^{(N-1) \wedge \tau_c} \left. \frac{\Delta \pi}{\Delta S} \right|_{S_j} R^{-j} \middle| S_k = s \right\}.
\end{aligned}$$

Alternatively, it is possible to obtain a pricing formula corresponding to Proposition 3.1. For $S \geq c$ define

$$\beta(S) = \frac{(R-d)S - c(R-1)}{(u-d)S} \tag{4.3}$$

and let $Q^{(c)}$ be the change of measure under which the upward increments evolve with probability law β and the downward movement with probability $1-\beta$ in the CRR model. We first check that indeed $Q^{(c)}$ is a probability measure.

Lemma 4.1 *If $S \geq c$ and $d < 1 < R < u$,*

$$0 < \beta(S) < 1.$$

Proof.

$$\frac{(R-d)S - c(R-1)}{(u-d)S} = \frac{R(S-c) + (c-dS)}{(u-d)S} > \frac{S-c+c-dS}{(u-d)S} > 0,$$

since $R > 1 > d$.

Also, since $1 < R < u$,

$$\frac{R(S-c)+(c-dS)}{(u-d)S} < \frac{u(S-c)+(c-dS)}{(u-d)S} = 1 - \frac{c(u-1)}{S(u-d)} < 1.$$

∎

We now have

Proposition 4.2 *Assume $d < 1 < R < u$ and the self-financing condition (4.1). For $S_k \geq c$ and $k \leq N$ define $M^{(c)}(k) = \pi(k, S_k)R^{-k}$. Then $M^{(c)}(k \wedge \tau_c)$, and $R^{-k}(S_{k\wedge\tau_c} - c)$, $k \leq N$ are $Q^{(c)}$ martingales.*

Proof. Since

$$\begin{aligned}
\mathbf{E}_{Q^{(c)}}\left\{M^{(c)}(k\wedge\tau_c)|\mathcal{F}_{k-1}\right\} &= \mathbf{E}_{Q^{(c)}}\left\{M^{(c)}(\tau_c)\mathbf{1}[\tau_c < k]|\mathcal{F}_{k-1}\right\} \\
&\quad + \mathbf{E}_{Q^{(c)}}\left\{M^{(c)}(k)\mathbf{1}[\tau_c \geq k]|\mathcal{F}_{k-1}\right\} \\
&= M^{(c)}(\tau_c)\mathbf{1}[\tau_c < k] \\
&\quad + \mathbf{1}[\tau_c \geq k]\mathbf{E}_{Q^{(c)}}\left\{M^{(c)}(k)|\mathcal{F}_{k-1}\right\}
\end{aligned}$$

it suffices to show that on $\tau_c \geq k, k \leq N$,

$$\mathbf{E}_{Q^{(c)}}\left\{M^{(c)}(j)|\mathcal{F}_{j-1}\right\} = M^{(c)}(j-1).$$

Indeed, using (4.3)

$$\begin{aligned}
\mathbf{E}_{Q^{(c)}}\left\{\pi(j,S_j)R^{-j}|\mathcal{F}_{j-1}\right\} &= R^{-j}\beta(S_{j-1})\varphi_{j-1}(uS_{j-1}-c) \\
&\quad +R^{-j}(1-\beta(S_{j-1}))\varphi_{j-1}(dS_{j-1}-c) \\
&\quad +R^{-(j-1)}\psi_{j-1}B_{j-1} \\
&= R^{-j}\varphi_{j-1}\left[\frac{R-d}{u-d}uS_{j-1}+\frac{u-R}{u-d}dS_{j-1}\right] \\
&\quad +R^{-j}\varphi_{j-1}\left[-c\frac{R-1}{(u-d)S_{j-1}}uS_{j-1}\right. \\
&\quad \left. +\,c\frac{R-1}{(u-d)S_{j-1}}dS_{j-1}-c\right] \\
&\quad +R^{-(j-1)}\psi_{j-1}B_{j-1} \\
&= R^{-j}\varphi_{j-1}\left[RS_{j-1}-c(R-1)-c\right] \\
&\quad +R^{-(j-1)}\psi_{j-1}B_{j-1} \\
&= R^{-(j-1)}\left[\varphi_{j-1}(S_{j-1}-c)+\psi_{j-1}B_{j-1}\right]
\end{aligned}$$

as claimed. To obtain that $R^{-k}(S_{k\wedge\tau_c}-c)$ is also a martingale, simply take $\varphi\equiv 1,\psi\equiv 0$. ∎

Corollary 4.2 *Under the assumptions of Proposition 4.2 and for $s\geq c, k\leq N$,*

$$\pi(k,s)=\mathbf{E}_{Q^{(c)}}\left[R^{-(N\wedge\tau_c-k\wedge\tau_c)}\tilde{V}\left(N\wedge\tau_c,S_{N\wedge\tau_c}\right)|S_k=s\right]$$

Thus, as in the continuous case (Section 3), using the Harrison-Pliska theory it follows from this representation that the price is arbitrage free on $[0,\tau_c\wedge N]$; one may check that each discrete sample path has positive probability under both probability laws $Q^{(0)},Q^{(c)}$. We remark, however, that the pricing formula given in Corollary 4.2 is numerically impractical since it requires the evaluation of probabilities at each price value. Instead, the formula in Corollary 4.1 involves probabilities that are known at the outset, thus simplifying the pricing of the option. This latter valuation was implemented in Burnes et al (1999).

5 Conclusions

As illustrated by the pricing of forest leases offered by the federal government, an adaptation of the Harrison-Pliska theory is required in order to quantify the effect of the "conversion cost," c specified in these contracts. The results presented in this paper apply to storable assets which due to the conversion cost may take non-positive values. The observation that the time for the developed asset to reach c is a stopping time, leads to a localization of the martingale theory. A striking consequence is that in addition to determining boundary conditions, the conversion cost appears explicitly in the governing partial differential equation.

The solution of this cost modified equation naturally leads to an Asian type option combined with a down and out barrier. The Asian characteristics of the solution are a direct consequence of the cost correction. The solution is discussed from two points of view. In one the solution is the discounted expected value of the the payoff under an equivalent local martingale measure. This representation has the advantage of establishing the lack of arbitrage opportunities up to the time the contracts are settled. For the second, we note that the formula for the expectation under Q leads to a representation of the price that is more suited for numerical calculations. The results of the computations are available in Burnes et al (1999).

In the familiar arbitrage free theory of valuations of options on traded securities, e.g. stocks, it is well-known that it is possible to use futures on the underlying asset as a hedge. For the contracts considered here one has available hedges using the developed asset or the undeveloped asset which may in general lead to different prices. Our approach provides a valuation of contracts using futures on the undeveloped asset where the key parameter is the *conversion cost*. By contrast, Morck etal (1989) have considered valuations from the standpoint of hedging by futures on the developed asset. The key parameter required by their approach is the *convenience yield,* which is defined by a yield furnished to the holder of the asset but not to the holder of the contract such as dividend in a preferred stock. As discussed by Morck etal (p 487, 1989) and Banks (p 236-240, 2000) in applications to natural resource valuations the identification of this parameter can be much more illusive. Comparison of these various pricing strategies may be useful as this theory is developed in this area.

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