THE DIRICHLET TO NEUMANN MAP - AN APPLICATION TO THE STOKES PROBLEM IN HALF SPACE

IHSANE BIKRI, RONALD B. GUENTHER AND ENRIQUE A. THOMANN

Department of Mathematics
Oregon State University
Corvallis, OR 97331, USA

Abstract. We illustrate the use of the Dirichlet to Neumann map for elliptic and parabolic problems in the context of the Stokes problems. An analogous representation to that obtained by Solonnikov in [5] for the case of a sphere is given for the half space problem. The validity of this representation is obtained establishing properties of the DtN map for the Laplace and Heat operators.

1. Introduction. The Dirichlet to Neumann (DtN) map is a common tool in the analysis of inverse problems in electrical exploration and in particular in impedance tomography, see for example the recent textbook [2] for a modern treatment. In these problems one is interested in determining the conductivity of a media knowing the electric field on the boundary resulting from the application of a known voltage.

The purpose of this paper is to illustrate the use of this notion in the study of solutions of boundary value problems for the Stokes problem of fluid mechanics, see eg [1] for a more complete treatment. We are particularly pleased to present these results in this special volume in honor of Professor V A Solonnikov whose work instigated this research.

In a series of papers starting with [4], and in the more recent [5], [6], [7], Professor Solonnikov has treated the problem of finding a solution to the time dependent Stokes problem

\[
\frac{\partial \mathbf{v}}{\partial t} - \Delta \mathbf{v} + \nabla p = 0, \quad \nabla \cdot \mathbf{v} = 0, \quad \mathbf{x} \in D, \ t > 0
\]

\[
\mathbf{v}|_{t=0} = 0
\]

\[
\mathbf{v}|_{x \in B} = \mathbf{a}(\mathbf{x}, t), \ t > 0.
\]

Here \( D \subset \mathbb{R}^3 \) is a domain, \( B \) is its boundary and \( \mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 \). The unknowns \( \mathbf{v} = \mathbf{v}(\mathbf{x}, t) = (v_1, v_2, v_3) \) is the velocity field and \( p = p(\mathbf{x}, t) \) is the scalar pressure whereas \( \Delta \) and \( \nabla \) denote the Laplacian and gradient operators calculated with respect to the spatial variables. It is also assumed from the outset, as done in [5], that the vector field \( \mathbf{a}(\mathbf{x}, t) \) satisfies the pointwise condition

\[
\mathbf{a}(\mathbf{x}, t) \cdot \mathbf{n}(\mathbf{x}) = 0, \ \mathbf{x} \in B
\]

where \( \mathbf{n}(\mathbf{x}) \) denotes the exterior unit normal to \( B \) at \( \mathbf{x} \).

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The solution constructed by Solonnikov in [5] is of the form
\[ v = w + \nabla V \]  
where \( w \) satisfies the (vector) heat equation
\[ \frac{\partial w}{\partial t} = \Delta w, \ w \in D, t > 0. \]  
The relation between \( p \) and \( V \) can easily be established by noting that
\[ -\nabla p = \frac{\partial v}{\partial t} - \Delta v = \frac{\partial w}{\partial t} - \Delta w + \nabla \left( \frac{\partial V}{\partial t} - \Delta V \right) \]
so
\[ p = -\left( \frac{\partial V}{\partial t} - \Delta V \right), \ x \in D, t > 0. \]
To determine \( w \) and \( V \), initial and boundary conditions are needed. For this purpose, Solonnikov introduces a scalar valued function \( Q(x, t) \) satisfying
\[ \Delta Q = 0, \ x \in D, \]  
\[ \frac{\partial Q}{\partial n} = \nabla' \cdot a'(x, t), \ x \in B \]  
where \( \nabla' \cdot \) denotes the tangential divergence of the tangential component of \( a \) to the boundary \( B \).

The Neumann problem (6) for \( Q \) has a solution since in light of (3) the solvability condition \( \int_B (a \cdot n) dS(x) = 0 \) is trivially satisfied. This solution is unique in the case that \( D \) is an unbounded domain requiring appropriate conditions at infinity, whereas for bounded domains, the uniqueness can be restored by requiring \( Q(x_0) = 0 \) for a fixed \( x_0 \in D \).

Having determined \( Q \), \( w \) is determined as the solution of the heat equation (5) subject to the boundary condition
\[ w(x, t) = a(x, t) + n(x, t)Q(x, t), \ x \in B \]  
and with homogeneous initial condition \( w(x, 0) = 0 \). Finally, \( V \) is determined so that the incompressibility condition is satisfied, i.e.
\[ \Delta V = -\nabla \cdot w, \ x \in D, t > 0, \]  
with homogeneous Dirichlet boundary condition
\[ V(x, t) = 0, \ x \in B, t > 0. \]  

From the construction above, it is apparent that \( v = w + \nabla V, p = -\partial V/\partial t + \Delta V \) is a solution of the Stokes equation (1). In order to check that the boundary condition (2) also holds, Solomnikov shows that in the case that \( D \) is the interior or the exterior of a sphere in \( \mathbb{R}^3 \), \( w, Q, V \) satisfying respectively (5), (6) and (9) with boundary data (8), (7) and (10) also satisfies
\[ \left( w \cdot n + \frac{\partial V}{\partial n} \right)_{|x \in B} = 0, \ t > 0. \]

In this paper, we consider the case where \( D = \mathbb{R}^3_+ = \{ x \in \mathbb{R}^3 : x_3 > 0 \} \) is the half space. In the course of the presentation, we will verify Solomnikov's assertions. The main emphasis is on fleshing out the structure of his approach for this special domain and in particular to illustrate the natural occurrence of the Dirichlet to Neumann map, mapping Dirichlet data on the boundary to its normal derivative and vice versa, for both the heat and Laplace operators.
Before proceeding, we introduce some well-known notation that will be used throughout the paper. First we define the Laplace transform of a function \(f(t)\) with \(t > 0\), to be
\[
\mathcal{L}_t(f(t))|_p = \int_0^\infty e^{-pt} f(t) dt.
\]
Similarly we introduce the analogous Laplace transform of a function \(f(x_3)\) defined for \(x_3 > 0\) given by
\[
\mathcal{L}_{x_3}(f(x_3))|_s = \int_0^\infty e^{-sx_3} f(x_3) dx_3.
\]
Finally, denote by \(\hat{f}(\xi)\) the Fourier transform of a function \(f : \mathbb{R}^2 \to \mathbb{R}\) defined as
\[
\hat{f}(\xi) = \int_{\mathbb{R}^2} e^{-i(x'\cdot\xi)} f(x') dx'.
\]
Here we denote \(x' = (x_1, x_2)\) points in \(\mathbb{R}^2\) and we will identify the points \((x_1, x_2)\) and \((x_1, x_2, 0)\) and write \(x'\) in both cases. Since we shall be working in the spatial half space \(\mathbb{R}^3_+\), the boundary \(\overline{D}\) is the plane \(x_3 = 0\) with exterior normal \(n = (0, 0, -1)\). Thus the condition (3) reduces to \(a_3(x') = 0\) and
\[
\nabla' \cdot a(x') = \frac{\partial a_1}{\partial x_1} + \frac{\partial a_2}{\partial x_2}.
\]
Finally, we denote by \(C^n.m(D \times (0, \infty))\) the space of functions that are \(n\) times continuously differentiable with respect to \(x \in D\) and \(m\) times continuously differentiable with respect to \(t\). In most cases, the domain \(D\) is either \(\mathbb{R}^2\) or \(\mathbb{R}^3_+\).

2. The Dirichlet to Neumann map for the Laplace and heat operator. In this section we obtain some basic results on the Dirichlet to Neumann map for the Laplace and heat operator in the half space domain \(\mathbb{R}^3_+\). We limit our presentation to those results that will be useful in the context of the Stokes problem. However some of the identities obtained here admit generalizations to different domains which we do not pursue in this paper.

We will utilize repeatedly the Fourier transform in \(\mathbb{R}^2\) together with the Laplace transform with respect to \(x_3\) of a function defined in \(\mathbb{R}^3_+\). Furthermore, since these transformations are applied to functions that also depend on time, we use the notation introduced in the previous section to obtain
\[
\mathcal{L}_{x_3}(\hat{f}(\xi, x_3, t))|_s = \int_0^\infty \int_{\mathbb{R}^2} f(x', x_3, t) e^{-i(x'\cdot\xi)} e^{-sx_3} dx'dx_3.
\]
We refer to this transform as the Laplace-Fourier transform of \(f(x, t)\).

Consider first the Laplace operator.

**Proposition 1.** Assume that \(\phi(x) \in C^2(\mathbb{R}^3_+)\) where it satisfies \(\Delta \phi = g(x)\). Then the Laplace-Fourier transform of \(\phi\) satisfies
\[
\mathcal{L}_{x_3} (\hat{\phi}(\xi, x_3))|_{|\xi|} = -|\xi| \hat{\phi}(\xi, x_3)|_{x_3=0} - \frac{\partial \hat{\phi}(\xi, x_3)}{\partial x_3}\bigg|_{x_3=0}.
\]

**Proof.** The Fourier transform of \(\phi\) satisfies
\[
-|\xi|^2 \hat{\phi}(\xi, x_3) + \frac{\partial^2 \hat{\phi}(\xi, x_3)}{\partial x_3^2} = \hat{g}(\xi, x_3).
\]
The Laplace transform of this equation with respect to $x_3$ is given by
\[
(-|\xi|^2 + s^2) \mathcal{L}_{x_3} \left(\hat{\phi}(\xi, x_3)\right) - s\hat{\phi}(\xi, x_3) = \frac{\partial \hat{\phi}(\xi, x_3)}{\partial x_3} + \mathcal{L}_{x_3} (\hat{g}(\xi, x_3))_x
\]
The theorem follows by taking $s = |\xi|$.

Proposition 1 allows us to relate the right hand side of the Laplace operator with the Dirichlet and Neumann data on the boundary. As an example one has the following result for the heat operator.

**Proposition 2.** Assume $\psi \in \mathcal{C}^{2,1}(R^d \times (0, \infty))$ and that it satisfies the heat equation $\partial \psi / \partial t = \Delta \psi$ in $R^d \times (0, \infty)$. Then, the Laplace-Fourier transform of $\psi$ satisfies
\[
\frac{\partial}{\partial t} \left( \mathcal{L}_{x_3} \hat{\psi}(\xi, x_3, t) \right)_{\xi} = -|\xi|\hat{\psi}(\xi, x_3, t)_{\xi} - \frac{\partial \hat{\psi}(\xi, x_3, t)}{\partial x_3}_{\xi} = 0
\]

**Proof:** This follows immediately from Proposition 1 by taking $g(x) = \hat{\psi}(x, t)$ and regarding $t$ as a parameter.

It is possible to give a representation of the Dirichlet to Neumann map for the heat operator in terms of standard kernels and differential operators. In the following proposition
\[
k^{(d)}(x, t) = \left(\frac{1}{4\pi t}\right)^{d/2} \exp\left(-\frac{|x|^2}{4t}\right)
\]
is the fundamental solution of the heat equation in $R^d$.

**Theorem 2.1.** Assume that $g(x', t) \in \mathcal{C}^{2,1}(R^2 \times (0, \infty))$ and $g(x', 0) = 0$. Assume that $u(x, t) \in \mathcal{C}^{2,1}(R^d_+ \times (0, \infty))$ is a classical solution of
\[
\frac{\partial u}{\partial t} = \Delta u, \quad x' \in R^2, x_3 > 0, t > 0
\]
\[
u_{|x_3=0} = g, \quad u_{|t=0} = 0.
\]

Then
\[
\frac{\partial u}{\partial x_3}_{x_3=0} = \int_0^t \frac{1}{\sqrt{4\pi s}} \int_{R^2} k^{(2)}(x' - y', s) \left( \frac{\partial g(y', t-s)}{\partial s} + \Delta g(y', t-s) \right) dy'ds
\]

**Proof.** Let $G(x' - y', x_3, y_3, s) = k^{(3)}(x' - y', x_3 - y_3, s) - k^{(3)}(x' - y', x_3 + y_3, s)$ denote the Green’s function for the heat equation in the half space problem. Then
\[
u(x', x_3, t) = \int_0^t \int_{R^2} G_{y_3}(x', y', x_3, 0, s) g(y', t-s) dy'ds
\]
where
\[
G_{y_3}(x', y', x_3, 0, s) = (-2)k^{(2)}(x' - y', s) \frac{\partial k^{(1)}(x_3, s)}{\partial x_3}
\]
Since $x_3 > 0$, it is possible to exchange the order of integration and differentiation to obtain
\[
\frac{\partial u(x', x_3, t)}{\partial x_3} = (-2) \int_0^t \frac{\partial^2 k^{(1)}(x_3, s)}{\partial x_3^2} \int_{R^2} k^{(2)}(x' - y', s) g(y', t-s) dy'ds.
\]
Let \( H(x', s, t) = \int_{\mathbb{R}^2} k^{(2)}(x' - y', s) g(y', t - s) dy' \). Since the heat kernel \( k^{(1)}(x_3, s) \) satisfies the heat equation (in one spatial dimension), one has

\[
\frac{\partial u(x', x_3, t)}{\partial x_3} = (-2) \lim_{\epsilon \to 0} \int_{\epsilon}^{t} \frac{\partial k^{(1)}(x_3, s)}{\partial s} H(x', s, t) ds \\
= -2 \lim_{\epsilon \to 0} k^{(1)}(x_3, s) H(x', s, t) |_{s=\epsilon} \tag{11}
\]

\[
+2 \lim_{\epsilon \to 0} \int_{\epsilon}^{t} k^{(1)}(x_3, s) \frac{\partial H(x', s, t)}{\partial s} ds 
\tag{12}
\]

We show first that (11) vanishes. This follows since, by assumption, \( g(x', 0) = 0 \) and thus \( H(x', t, t) = 0 \). On the other hand, since \( x_3 > 0 \), it follows that \( \lim_{\epsilon \to 0} k^{(1)}(x_3, \epsilon) = 0 \).

In order to proceed, note that since \( s > 0 \) it is also possible to exchange the order of differentiation and integration once more to obtain that (12) can be written as

\[
\int_{\epsilon}^{t} k^{(1)}(x_3, s) \frac{\partial H(x', s, t)}{\partial s} ds \tag{13}
\]

Since \( k^{(2)} \) satisfies the heat equation, the time derivatives can be exchanged by spatial derivatives, and integrating by parts one has that (13) can be written as

\[
\int_{\epsilon}^{t} k^{(1)}(x_3, s) \int_{\mathbb{R}^2} k^{(2)}(x' - y', s) \frac{\partial g(y', t - s)}{\partial s} dy' ds \\
+ \int_{\epsilon}^{t} k^{(1)}(x_3, s) \int_{\mathbb{R}^2} \frac{\partial k^{(2)}(x' - y', s)}{\partial s} g(y', t - s) dy' ds 
\]

Thus, taking the limit as \( \epsilon \to 0 \) one has

\[
\frac{\partial u(x', x_3, t)}{\partial x_3} = 2 \int_{\epsilon}^{t} k^{(1)}(x_3, s) \int_{\mathbb{R}^2} k^{(2)}(x' - y', s) \left( \frac{\partial g(y', t - s)}{\partial s} + \Delta y' g(y', t - s) \right) dy' ds.
\]

The theorem follows by taking the limit as \( x_3 \to 0 \). □

Let’s denote by \( \text{HDtN} \) the Dirichlet to Neumann map corresponding to the heat equation in the half space domain, that is

\[
\text{HDtN}(g) = \frac{\partial u}{\partial x_3} |_{x_3=0}
\]

where \( u(x, t) \) is the solution of Dirichlet problem for the heat equation with boundary data \( g \), i.e.

\[
\frac{\partial u}{\partial t} = \Delta u, \quad x' \in \mathbb{R}^2, x_3 > 0, t > 0
\]

\[
u |_{x_3=0} = g, \quad u |_{t=0} = 0.
\]

Similarly, denote by \( \text{HNtD} \) the Neumann to Dirichlet map defined by \( \text{HNtD}(g)(x', t) = v(x', t) \), where \( v(x, t) \) is the solution of the Neumann problem for the heat equation.
with boundary data \( g \)

\[
\frac{\partial v}{\partial t} = \Delta v, \quad x' \in \mathbb{R}^2, x_3 > 0, t > 0
\]

\[
\left. \frac{\partial v}{\partial x_3} \right|_{x_3=0} = g, \quad v|_{t=0} = 0.
\]

The result of the previous theorem can be expressed in terms of these mappings as follows.

**Corollary 1.** Under the same assumption of Theorem 2.1, one has

\[
\left[ HDtN + \frac{\partial HNdD}{\partial t} \right](g)(x', t) = \int_0^t \frac{1}{\sqrt{\pi s}} \int_{\mathbb{R}^2} k^{(2)}(x' - y', s) \Delta g(y', t - s) dy' ds
\]

**Proof.** Recall that the fundamental solution of the Neumann problem in the half space \( \mathbb{R}^2_+ \) is given by \( N(x', x_3, y_3, s) = k^{(3)}(x', x_3 - y_3, s) + k^{(3)}(x', x_3 + y_3, s) \). Then the HNdD mapping is given by

\[
HNdD(g)(x', t) = \int_0^t \frac{1}{\sqrt{\pi s}} \int_{\mathbb{R}^2} k^{(2)}(x' - y', s) g(y', t - s) dy' ds.
\]

The Corollary follows by noting that

\[
\int_0^t \frac{1}{\sqrt{\pi s}} \int_{\mathbb{R}^2} k^{(2)}(x' - y', s) \frac{\partial g(y', t - s)}{\partial s} dy' ds
\]

\[
= - \int_0^t \frac{1}{\sqrt{\pi s}} \int_{\mathbb{R}^2} k^{(2)}(x' - y', s) \frac{\partial g(y', t - s)}{\partial t} dy' ds
\]

\[
= - \frac{\partial}{\partial t} \int_0^t \frac{1}{\sqrt{\pi s}} \int_{\mathbb{R}^2} k^{(2)}(x' - y', s) g(y', t - s) dy' ds
\]

In the last step, we have used once more the compatibility condition \( g(x', 0) = 0 \).

From Theorem 2.1 it is possible to compute the double Laplace (in time) Fourier transform of the HDtn mapping. We present this result first and then provide an independent proof.

**Theorem 2.2.** Under the same assumption of Theorem 2.1 one has

\[
\left. \frac{\partial}{\partial x_3} \left[ L_1(\hat{u}(\xi, x_3, t)) \right] \right|_{x_3=0} = -\sqrt{p + \|\xi\|^2} L_1(\hat{g}(\xi, t))
\]

**Proof.** Using the explicit form of the Green’s function in the half space case, one has

\[
u(x', x_3, t) = x_3 \int_0^t \int_{\mathbb{R}^2} \frac{1}{s} k^{(1)}(x_3, s) k^{(2)}(x' - y', s) g(y', t - s) dy' ds
\]

Recall that

\[
\hat{k}^{(2)}(\xi, t) = e^{-4\xi^2}
\]

so that the Fourier transform of the solution \( u \) is given by

\[
\hat{u}(\xi, x_3, t) = \frac{x_3}{\sqrt{4\pi}} \int_0^t \frac{e^{-|s|^2}}{s^{3/2}} e^{-|\xi|^2 s} \hat{g}(\xi, t - s) ds.
\]
The theorem follows taking the Laplace transform (in time) of this expression, once we show that

$$\frac{x_3}{4\pi} \int_0^\infty \frac{1}{s^{3/2}} \exp \left( -\frac{x_3^2}{4s} + \left( |\xi|^2 + p \right)s \right) ds = \exp \left(-\sqrt{p + |\xi|^2 x_3} \right).$$  \hspace{1cm} (14)

To check this, recall that the modified Bessel function of order $\nu$ is given by

$$K_\nu(z) = \frac{1}{2} \int_0^\infty t^{-(1+\nu)} \exp \left(- \frac{z}{2} \left( \frac{1}{t} + t \right) \right) dt$$

and that in particular

$$K_{1/2}(z) = \left( \frac{\pi}{2z} \right)^{1/2} e^{-z}.$$ \hspace{1cm} (15)

See Magnus, Oberhettinger and Soni [3] for basic properties of these functions.

The substitution

$$t = s \frac{2\sqrt{|\xi|^2 + p}}{x_3}$$

transforms the left hand side of (14) into

$$\frac{x_3}{4\pi} \left( \frac{2\sqrt{|\xi|^2 + p}}{x_3} \right)^{1/2} \int_0^\infty \frac{1}{t^{3/2}} \exp \left(- \frac{x_3}{2} \sqrt{\frac{|\xi|^2 + p}{t}} \right) \frac{1}{t} dt$$

$$= \left( \frac{2x_3}{\sqrt{\pi}} \right)^{1/2} \frac{\sqrt{\pi}}{K_{1/2} \left(x_3 \sqrt{|\xi|^2 + p} \right)} \frac{x_3}{2} \sqrt{\frac{|\xi|^2 + p}{t}} \exp \left(-x_3 \sqrt{\frac{|\xi|^2 + p}{t}} \right)$$

using the explicit form (15).

**Remark 1.** A different proof of Theorem 2.2 can be obtained using the Laplace transform (with respect to time) of the Fourier transform of the heat equation in the half space domain $R^3_+$. Indeed, such a transform clearly satisfies

$$\frac{\partial^2}{\partial x_3^2} \left[ L_t(\tilde{u}(\xi, x_3, t)) \right] = (p + |\xi|^2) L_t(\tilde{u}(\xi, x_3, t))$$

whose solution that decays as $x_3 \to \infty$ is given by

$$L_t(\tilde{u}(\xi, x_3, t)) = L_t(\tilde{g}(\xi, t)) \exp \left(-x_3 \sqrt{|\xi|^2 + p} \right)$$

The theorem follows by direct differentiation. The longer proof presented has the advantage of giving an explicit formula for the different kernels involved in the calculation.

3. **Application to the Stokes problem.** In this section the explicit representation and calculations of the Dirichlet to Neumann for the Laplace and heat operator are used in the context of the Stokes problem in the half space problem. Recall here that the solution of the Stokes problem is sought in the form $v = w + \nabla V$, where

$$\Delta V = -\nabla \cdot w, \text{ in } R^3_+, \quad V = 0 \text{ on } x_3 = 0$$ \hspace{1cm} (16)

$$\frac{\partial w}{\partial t} = \Delta w \text{ in } R^3_+ \times (0, \infty), \quad w(x, 0) = 0, \quad w(x', 0, t) = a(x', t) + nQ$$ \hspace{1cm} (17)

$$\Delta Q = 0 \text{ in } R^3_+, \quad \frac{\partial Q}{\partial n} = \nabla' \cdot a'(x', t) \text{ on } x_3 = 0.$$ \hspace{1cm} (18)
Recall also, that only the validity of the boundary condition remains to be established, that is, one needs to check that on \( x_3 = 0 \), \( \mathbf{v}(x', t) = \mathbf{a}(x', t) \). It is apparent that the tangential components agree, that is \( v_j(x', t) = a_j(x', t), j = 1, 2 \), in view of the boundary conditions imposed on \( V \) and \( \mathbf{w} \). Furthermore, since on \( x_3 = 0 \), \( w_3 = -Q \), it remains only to check that

\[
Q(x', t) - \frac{\partial V}{\partial x_3} \bigg|_{x_3=0} = 0.
\]

In checking this identity the various results from the previous section will be utilized with specific choices of data.

We first note a representation for the boundary values of \( Q \).

**Lemma 3.1.** Assume that \( \mathbf{a}(x', t) \in C^{1,1}(\mathbb{R}^2 \times (0, \infty)) \). Then the Laplace transform of \( Q(\mathbf{x}, t) \) satisfying (18) is given by

\[
|\xi| \hat{Q}(\xi, x_3, t) \bigg|_{x_3=0} = \nabla' \cdot \mathbf{a}'(\xi, t)
\]

**Proof.** This is an immediate consequence of Proposition 1 with \( \phi = Q, g = 0 \) and noting that the boundary condition gives

\[
\frac{\partial Q}{\partial x_3} = -\nabla' \cdot \mathbf{a}'(x', t).
\]

Relating the normal derivative of \( V \) with the boundary data is more involved and it involves the divergence of the vector field \( \mathbf{w} \) as the following results shows.

**Proposition 3.** Assume that \( \mathbf{w} \in (C^{1,1}(\mathbb{R}^3_+ \times (0, \infty)))^3 \) satisfies (17) and let \( \rho = \nabla \cdot \mathbf{w} \). Then the Fourier transform of \( V \) satisfying (16) satisfies

\[
\frac{\partial}{\partial t} \left( \frac{\partial \hat{V}}{\partial x_3} \bigg|_{x_3=0} \right) = - |\xi| \hat{\rho}(\xi, x_3, t) \bigg|_{x_3=0} - \frac{\partial \hat{\rho}}{\partial x_3} \bigg|_{x_3=0}
\]

**Proof.** Note that from equation 16 and Proposition 1 one has

\[
\frac{\partial \hat{V}(\xi, x_3, t)}{\partial x_3} \bigg|_{x_3=0} = \mathcal{L}_x(\hat{\rho}(\xi, 0, t))(|\xi|)
\]

Consequently their time derivatives agree. The result follows applying Proposition 2 with \( \psi = \rho \).

The following lemmas provide the relation between \( Q \) and the divergence of the vector field \( \mathbf{w} \).

**Lemma 3.2.** Assume that \( \mathbf{a} \in C^{3,1}(\mathbb{R}^2 \times (0, \infty)) \) and that \( \mathbf{w} \) and \( Q \) satisfy (17) and (18) respectively. Then, with \( \rho = \nabla \cdot \mathbf{w} \),

\[
\mathcal{L}_t (\hat{\rho}(\xi, 0, t)) \big|_{\rho} = \left( |\xi| + \sqrt{p + |\xi|^2} \right) \mathcal{L}_t \left( \hat{Q}(\xi, 0, t) \right) \big|_{\rho}
\]

**Proof.** Note that on \( x_3 = 0 \),

\[
\rho(x', t) = \nabla' \cdot \mathbf{a}' + \frac{\partial w_3}{\partial x_3} \bigg|_{x_3=0}
\]
the Laplace transform of this time derivate is given by

\[ \mathcal{L}_t(\dot{\rho}(\xi, 0, t)) |_{p} = \mathcal{L}_t\left(\nabla' \cdot \mathbf{a}\right) |_{p} + \mathcal{L}_t\left(\frac{\partial \hat{w}_3}{\partial x_3} |_{x_3=0}\right) |_{p} \]

\[ = \mathcal{L}_t\left(\nabla' \cdot \mathbf{a}\right) |_{p} + \frac{\partial \mathcal{L}_t(\hat{w}_3)|_{p}}{\partial x_3} |_{x_3=0} \]

Using Lemma 3.1, the first term is given by

\[ |\xi| \mathcal{L}_t\left(\hat{Q}(\xi, 0, t)\right) |_{p} \]

Since \( w_3(x', 0, t) = -\hat{Q}(x', t) \) an application of Theorem 2 gives

\[ \frac{\partial \mathcal{L}_t(\hat{w}_3)}{\partial x_3} |_{x_3=0} = \sqrt{p + |\xi|^2} \mathcal{L}_t\left(\hat{Q}(\xi, 0, t)\right) |_{p} . \]

\[ \square \]

**Lemma 3.3.** Under the same assumptions of Lemma 3.2 one has

\[ \mathcal{L}_t\left(\frac{\partial \hat{p}(\xi, x_3, t)}{\partial x_3} |_{x_3=0}\right) |_{p} = -\sqrt{p + |\xi|^2} \left(|\xi| + \sqrt{p + |\xi|^2}\right) \mathcal{L}_t\left(\hat{Q}(\xi, 0, t)\right) |_{p} \]

**Proof.** Note that

\[ \mathcal{L}_t\left(\frac{\partial \hat{p}(\xi, x_3, t)}{\partial x_3} |_{x_3=0}\right) |_{p} = \frac{\partial [\mathcal{L}_t(\hat{p}(\xi, x_3, t))] |_{p}}{\partial x_3} |_{x_3=0} \]

\[ = -\sqrt{p + |\xi|^2} \mathcal{L}_t(\hat{p}(\xi, 0, t)) |_{p} \]

\[ = -\sqrt{p + |\xi|^2} \left(|\xi| + \sqrt{p + |\xi|^2}\right) \mathcal{L}_t(\hat{Q}(\xi, 0, t)) |_{p} \]

where in the last steps, Theorem 2.2 and Lemma 3.2 were used. \( \square \)

The main result is a consequence of these lemmas.

**Theorem 3.4.** Assume that \( \mathbf{a} \in C^{3,1}(R^2 \times (0, \infty)) \) satisfies the compatibility condition \( \mathbf{a}(x', 0) = 0 \), and \( a_3(x', t) = 0 \). Let \( V, \mathbf{w} \) and \( Q \) be solutions of (16), (17) and (18) respectively. Then \( \mathbf{v} = \mathbf{w} + \nabla V, p = -\partial V/\partial t + \Delta V \) is a solution of the Stokes problem (1) with boundary condition \( \mathbf{v}(x', t) = \mathbf{a}(x', t) \)

**Proof.** It only remains to show that

\[ Q(x', t) - \frac{\partial V}{\partial x_3} |_{x_3=0} = 0. \]

Note that from the compatibility conditions imposed on \( \mathbf{a} \) this relation holds at \( t = 0 \). So, it is sufficient to show that the time derivative of the above expression vanishes for \( t > 0 \) or equivalently its Laplace transform. Proposition 3 shows that the Laplace transform of this time derivate is given by

\[ p \mathcal{L}_t\left(\hat{Q}(\xi, 0, t)\right) |_{p} + |\xi| \mathcal{L}_t(\hat{p}(\xi, 0, t)) |_{p} + \mathcal{L}_t\left(\frac{\partial \hat{p}(\xi, x_3, t)}{\partial x_3} |_{x_3=0}\right) |_{p} . \]
Lemmas 3.2 and 3.3 express the last two terms in the above expression in terms of $\mathcal{L}_t \left( \overline{Q}(\xi, 0, t) \right)$. It is a routine calculation to check that the expression indeed vanishes.

4. **Concluding remarks.** In most cases, it is impossible to find explicit solutions for the classical problems of mathematical physics except in the case of special geometries such as spheres, rectangles, half spaces and a few other domains. It is instructive, however, to look closely at the structure of the solutions where such solutions are obtainable to discern their relevant properties, and to explore the underlying relationships between the components that make up the solution. In the present paper, we followed the lead of Professor Solonnikov and chose a half space domain. Using the Dirichlet to Neumann map for the Laplace and heat operator, we establish relations between the underlying solution to elliptic and parabolic problems and the Stokes problem. It is expected that similar analysis can be carried out in other geometries. The simple domain considered here allows us to utilize Fourier methods which simplifies the calculations and analysis. It is natural to speculate that when one considers other geometries, the use of Fourier methods will be replaced by eigenfunction expansion for the Laplace and heat operator. The Stokes problem serves as motivation for exploring relations between the Dirichlet to Neumann for these operators in more general geometries. The work of Professor Solonnikov has opened up this intriguing line of research.

**REFERENCES**


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E-mail address: bikri@math.orst.edu
E-mail address: guenth@math.orst.edu
E-mail address: thomann@math.orst.edu