The Various Valuation Methods for Non-Attainable Contingent Claim

Hyungho Youn

Oregon State University

Jan, 2004

1. Introduction

Black and Scholes (1973) derived the option price of a stock with geometric Brownian motion by constructing a hedging portfolio. Harrison and Pliska (1981) explicitly explained a number of fundamental concepts for understanding option pricing such as arbitrage-free, self-financing and replicating portfolio, attainability, and completeness. Harrison and Pliska (1981) showed that if a contingent claim is attainable in no-arbitrage market, option price is easily computed using an equivalent martingale. In fact, Cox, Ross, and Rubinstein (1979) already used the concept of self-financing and replicating portfolio in multi-period binomial model. The fundamental contribution of Harrison and Pliska (1981) is, however, to tie this self-financing and replicating portfolio approach to equivalent martingale measure.

Cox, Ross, and Rubinstein (1979) used a multiple-period binomial model, and showed that the model can be one for geometric Brownian motion or Poisson process with fixed size jump in the limit by the law of central event and the rare event respectively. In fact, the multi-period binomial model is important in option pricing because any contingent claim in the model is attainable by Binomial Representation Theorem. Thus, any contingent claim in the stock market with geometric Brownian motion or Poisson process with fixed size is attainable.

However, Press (1967) noted long ago that the mixture of geometric Brownian motion and Poisson process agrees with the empirical stock price movement. A problem with this mixture process is that there may exist a non-attainable contingent claim in a stock market defined by the process. Merton (1976) had a stock price evolve with both geometric Brownian motion and Poisson process with varying jumps. He showed it is impossible to construct a hedging portfolio of stock and option in the particular market as Black and Scholes did for geometric Brownian motion. Naik and Lee (1990) showed an example of contingent claim that cannot be replicated when the stock price follows the mixture process.

When stock price movement can follow the mixture process, we neither use a risk neutral valuation (i.e. equivalent martingale measure) nor construct a hedging portfolio in pricing a non-attainable contingent claim. Thus, a question arises as to how a non-attainable contingent claim should be valued. We examine various methods such as Merton's method (1976), utility maximization (Naik and Lee 1990, Davis 1994), Esscher measure (Gerber and Shiu 1994), and a variance minimizing hedging (Schweiser 1992, 1995). Also, we introduce a new valuation method. This paper compares these methods in a one period model.

2. The Example of Non-attainable Contingent Claim

Harrison and Pliska (1981) derived a valuation formula for an attainable contingent claim on the assumption that (i) asset prices must be positive, (ii) there must be a riskless bond, and (iii) there is no transaction cost. Using this formula, the famous Black-Scholes option price formula (1973) is easily derived.

Proposition 1 (**Risk Neutral Valuation**, Harrison and Pliska 1981)
Within the framework defined above, the value of *any attainable* contingent claim X at maturity T is given by:

$$\pi(t) = R^{-(T-t)} E^*(X \mid F_t)$$

where E^* is the expectation with respect to an equivalent martingale measure and $R = 1 + r$ is risk free yield with r a risk free interest. [1]

However, there can be a non-attainable contingent claims which cannot be replicated by self-financing trading strategies. Merton (1976) introduced a stock price path with the mixture of geometric Brownian motion and Poisson jump process

[1] Harrison and Pliska (1981) also showed there exist equivalent martingale measures in arbitrage-free market. It is believed that the market is competitive such that no arbitrage is allowed.

with varying sizes. Then, he showed that no portfolio consisting of the stock and the corresponding option hedges risk. It implies indirectly that Merton's model is incomplete in the sense that there exists a non-attainable contingent claim. Naik and Lee (1990) showed directly an example of non-attainable contingent claim when a stock price evolves with the mixture of geometric Brownian motion and a fixed size jump[2]. First, we examine a one period model where a jump can occur, and then the Merton (1976), and the Naik and Lee (1990) models, respectively.

One period model

For simplicity we consider a model in which a stock can take three possible values, *u* and *d*, both regular movements with probability of 0.59 and 0.39 respectively, and *j*, an unexpected rare jump with a small probability of 0.02. A contingent claim (a_1, a_2, a_3) is not necessarily replicated by a portfolio of the stock and the riskless bond. Since a contingent claim space is R^3, a stock value vector (u, d, j) and a bond value vector $(1+r, 1+r, 1+r)$ cannot span the contingent claim space.

Merton's model (1976)

Following Merton (1976) we show that the construction of hedging portfolio is, however, not possible when the returns of stock contain jumps of random and unpredictable sizes occurring at random times.

The stock price returns is written as a stochastic differential equation as:

$$dS / S = (\alpha - \lambda k)dt + \sigma dB + (Y-1)dN \qquad (1)$$

[2] The stock price evolution path with the mixture of geometric Brownian motion and a fixed size jump is enough to have incomplete market.

where α and σ^2 are the instantaneous expected return and variance of Brownian motion; dB is a standard Brownian motion, $N(t)$ is the independent Poisson process; dN and dB are assumed to be independent; λ is the mean number of arrivals per unit time; $k = E(Y-1)$ where $(Y-1)$ is the random variable percentage change in stock price if the Poisson event occurs.

If α , λ , k, and σ are constants, then the stock price at time t (assuming $S(0)=1$) can be written as:

$$S(t) = \exp[(\alpha - \sigma^2 / 2 - \lambda k)t + \sigma B(t)] \prod_{j=1}^{n} Y_j \tag{2}$$

where $B(t)$ is a Brownian motion, the Y_j are independently and identically distributed, and $N(t) = n$ is a Poisson distributed with parameter λt.

We now turn to the dynamics of the option price. Suppose that the option price, W, can be written as a function of the stock price and time: namely $W(t) = F(S,t)$. Then, the option return dynamics is derived as [3]:

$$dW/W = (\alpha_W - \lambda k_W)dt + \sigma_W dB + dN_W .$$

where

$$\alpha_w = [(1/2)\sigma^2 S^2 F_{SS}(S,t) + (\alpha - \lambda k)F_S(S,t) + F_t + \lambda E\{F(SY,t) - F(S,t)\}] / F(S,t) \tag{3}$$

$$\sigma_w = F_S(S,t)\sigma / F(S,t) \tag{4}$$

$dN_W = Y_W - 1 = F(SY,t) - F(S,t)] / F(S,t)$ if a jump occurs.

Next, consider a portfolio strategy which holds the stock and the option in proportion w_1, and w_2 where $w_1 + w_2 = 1$. If P is the value of the portfolio, then the return dynamics of the portfolio can be written as:

---

[3] This is derived by using Ito's Lemma for Brownian motion and Poisson process.

$$dP/P = (\alpha_P - \lambda k_P)dt + \sigma_P dB + dN_P \tag{5}$$

where

$$\alpha_p = w_1\alpha + w_2\alpha_w \tag{6}$$

$$\sigma_p = w_1\sigma + w_2\sigma_w \tag{7}$$

$$dN_P = Y_p - 1 = w_1(Y-1) + w_2[F(SY,t) - F(S,t)]/F(S,t). \tag{8}$$

To see that there does not always exist a set of portfolio weight (w_1, w_2) that will have risk-free return after eliminating risk, simply observe that the number of the following equations exceeds the number of unknown w_1, and w_2 :

$$w_1\alpha + w_2\alpha_w = r \tag{9}$$

$$w_1\sigma + w_2\sigma_w = 0 \tag{10}$$

$$w_1(Y-1) + w_2[F(SY,t) - F(S,t)]/F(S,t) = 0 \tag{11}$$

Naik and Lee (1990)

The Naik and Lee (1990) model is similar to that of Merton (1976) except for fixed jump size. The path $x(t)$ of the logarithm of stock price is

$$\log\frac{S(t)}{S(0)} = x(t) = B(t) + n(t) = B(t) + N(t) - \lambda t\ . \tag{12}$$

where $n(t)$ is the compensated Poisson process.

Again using Ito's lemma, we obtain (assuming $S(0) = 1$):

$$S(t) = \exp[B(t) - (1/2)t - \lambda t + \log(2)N(t)]. \tag{13}$$

Note that this $S(t)$ is a martingale. Let $y(t) = B(t) + \log 2 \times n(t)$ be a contingent claim[4].

Claim (Naik and Lee 1990): This $y(t)$ cannot be written as a stochastic integral with respect to $x(t)$.

Proof: Suppose that $y(t)$ is replicated by $x(t)$, i.e. $y(t) = y(0) + \int_0^t \alpha(s)dx(s)$. Since $y(0) = B(0) + n(0) = 0$, $y(t) = \int_0^t \alpha(s)dx(s)$. Thus,

$y(t) = \int_0^t \alpha(s)dx(s) = \int_0^t \alpha(s)dB(s) + \int_0^t \alpha(s)dn(s)$ by (12), but it is also true that

$y(t) = 1\int_0^t dB(s) + \log 2\int_0^t dn(s)$ by the definition of the contingent claim. The comparison of two forms of $y(t)$ implies $\Pr ob(t : \alpha(t) = 1, \alpha(t) = \log 2) > 0$.

But it is a contradiction.

This will imply that $y(t)$ cannot be written as a stochastic integral with respect to $S(t)$ so that $y(t)$ is not attainable so the market is not complete.

We showed three examples where the markets are not complete. Harrison and Pliska (1981) found another important relation between equivalent martingale measure and market completeness.

Proposition 2 (Harrison and Pliska 1981)

Under the conditions stated at the beginning of this section, assume that there is at least one equivalent martingale measure. Then, the market is complete if and if the equivalent martingale measures is unique.

---

[4] This contingent claim $y(t)$ may be negative while Harrison and Pliska (1981) assumed that contingent claim is positive.

Proposition 2 implies that these examples of incomplete markets have multiple equivalent martingales. First, consider a one period model. To find equivalent martingale measures, we solve $ES(1) = 1 + r$, i.e.

$$(p_u u + p_d d + p_j j) = 1 + r \text{ and } p_u + p_d + p_j = 1.$$

There are many combination of p_u, p_d, and p_j that satisfy the two equations.

For both Merton (1976), and Naik and Lee (1990) models, we slightly modify Merton's model and define stock price path by:

$$dS / S = \mu dt + \sigma dB + (Y - 1)dN \tag{14}$$

where jump size $Y > 1$ is fixed.

The stock price at time *t* (assuming $S(0) = 1$) can be written as:

$$S(t) = \exp[(\mu - (\sigma^2 / 2))t + \sigma B(t)]Y^n \tag{15}$$

The Brownian motion and Poisson process are assumed to be independent so the expected stock price at time *t* is:

$$E(S(t)) = e^{\mu t} EY^n = e^{\mu t} \sum_{n=0}^{\infty} Y^n \frac{(\lambda t)^n e^{-\lambda t}}{n!} = e^{\mu t - \lambda t} \sum_{n=0}^{\infty} Y^n \frac{(\lambda t)^n}{n!} = e^{[(\lambda(Y-1)+\mu]t}. \tag{16}$$

.

We may simply change the distribution parameters in order to find equivalent martingale measures. The point is that there are many combinations of λ and μ (or many equivalent martingales measure) that makes $E(S(t)) = e^{rt}$ or $\lambda(Y - 1) + \mu = r$

Now we face a problem of how we deal with multiple equivalent martingale measures when the market is incomplete. First, even though the market is incomplete, some contingent claim may be attainable. Suppose that a contingent claim gives stock value at maturity T in the above model (incomplete market). Such a contingent claim is attainable by holding one unit of stock until maturity. For an attainable contingent claim, both Proposition 1 and 3 are true.

Proposition 3 (Harrison and Pliska 1981)
In the market model framework defined at the beginning of section 2, if the market is arbitrage free, then any attainable contingent claim is uniquely replicated.

If we combine Proposition 1 and 3, we can conclude that an attainable contingent claim even in the incomplete market can be valued and the risk neural valuation by any equivalent martingale is the same. The use of any combination of λ and μ (or any equivalent martingales measure) in the above model that makes $E(S) = e^{rt}$ yields the same contingent claim value. Later, we will show explicitly this holds for the one period model. However, if a contingent claim in an incomplete market is not attainable, then the question is which one we use out of multiple of equivalent martingale measures. This question is addressed in the next section.

3. **The Valuation of Non-Attainable Contingent Claim**

Merton's Model

Merton (1976) supposed that the jump component of the stock's return will represent non-systematic risk, i.e. the jump component will be uncorrelated with the market. Since jump process is compensated, one is risk-neutral about the jump process. Then, one is risk-conscious of only continuously occurring events represented by geometric Brownian motion, but risk neutral toward rarely occurring jumps. It is enough to solve only (9) and (10) to find option price.

$$\alpha_P = w_1\alpha + w_2\alpha_w = r \tag{17}$$

$$\sigma_P = w_1\sigma + w_2\sigma_w = 0 \tag{18}$$

The two equations are consolidated into

$$(\alpha - r)/\sigma = (\alpha_w - r)/\sigma_w. \tag{19}$$

By substituting (3) and (4) into (19), we obtain

$$(1/2)\sigma^2 S^2 F_{SS}(s,t) + (r - \lambda k)SF_S(S,t) + F_t + \lambda E\{F(SY,t) - F(S,t)\} = rF(S,t) \tag{20}$$

Define W to be the Black-Scholes option pricing formula at time 0 for the no-jump case when the current stock price is $S(0) = 1$ and maturity is T, i.e.

$$W(S(0)) = S(0)\Phi(d_1) + Ke^{-rT}\Phi(d_2)$$

where Φ is the cumulative normal distribution,

$d_1 = [\log(S(0)/K) + (r + \sigma^2/2)T]/\sigma\sqrt{T}$ and $d_2 = d_1 - \sigma\sqrt{T}$.

The solution to (20), i.e. the option price, can be written as

$$F(S(0)) = \sum_{n=0}^{\infty} \frac{e^{-\lambda T}(\lambda T)^n}{n!}[E_n\{W(\Pi_{j+1}^{n} Y_j e^{-\lambda kT})\}]. \tag{21}$$

We can reinterpret Merton’s formula in terms of Black and Scholes price as follows. We rearrange the stock price path (2) into:

$$S(t) = \Pi_{j=1}^{n} Y_j e^{-\lambda k t} e^{(\alpha - \sigma^2/2)t + \sigma B(t)} . \tag{22}$$

Then, the rearranged stock price path is one with an initial stock price of $\Pi_{j=1}^{n} Y_j e^{-\lambda k t}$ instead of $S(0) = 1$ given the number of jumps and jump size at each jump. Therefore, Merton's price formula (21) is the consecutive conditional expectations of Black-Schole price using the original probability parameters of the Poisson process:

$$\sum_{n=0}^{\infty} \frac{e^{-\lambda T} (\lambda T)^n}{n!} [E_n \{W(\Pi_{j+1}^{n} Y_j e^{-\lambda k T})\}]$$

$$= E\{E\ (BS\ price\ |each\ jump\ size)|the\ number\ of\ jump)\} . \tag{23}$$

Again, (23) implies that one is risk-conscious of only continuously occurring events represented by geometric Brownian motion, but risk neutral toward rarely occurring jumps so she uses the original probability parameters of the Poisson process [5].

Utility Maximization

Naik and Lee (1990) raised a question about Merton's assumption that the jumps in stock price are uncorrelated with the return on the market portfolio. Clearly, this assumption is violated if the stock under consideration is the market portfolio itself. Merton's formula is interpreted as an attempt to use an equivalent martingale measure conditional on the event that jumps occur for non-attainable contingent claim.

---

[5] One uses a risk-neutral probability (or an equivalent martingale measure) when she is risk-conscious of occurring events represented by geometric Brownian motion, while she uses the original probability when risk neutral toward rarely occurring.

In Von Neumann and Morgenstern (1944), a risk-averse agent used a utility function to value a contingent claim. Her utility of the contingent claim is calculated by a particular form of the utility function and the original probability measure (not an equivalent martingale measure). Lucas (1976) presented a continuous portfolio model where one constantly creates a self-financing portfolio to maximize utility. Then, utility maximization and market clearing determines the current stock price. Naik and Lee (1990) used Lucas' model to value a non-attainable contingent claim, but it is noted that their formula depends on the utility function and uses the original probability measure.

In the economy, there is one risk-averse agent and one firm. The firm issued one unit of stock and $S(t)$ is the stock price at time t. The utility function of the agent exhibits constant relative risk aversion. A convenient form is $U(c,t) = e^{-rt}\frac{c^{\gamma}}{\gamma}$ where c is consumption and $0 < \gamma < 1$ means the degree of risk aversion[6]. The firm's production is affected by a continuous small perturbation such as regular economic fluctuation as well as by major shocks such as technology breakthrough or disasters. As results, the firm's dividend follows the path [7]:

$$\delta(t) = \exp[(\alpha - \sigma^2/2 - \lambda k)t + \sigma B(t)]\exp\sum_{j=1}^{N(t)} Y_j \qquad (24)$$

The agent owns the company and her decision is whether to consume the dividend wholly or to save some of them for utility maximization. Market clearing dictates that she consume the dividend wholly in maximizing utility. The agent seeks to maximize the expected utility of the consumption stream:

[6] A small γ implies higher risk aversion.

[7] The equation (24) is different from (2) in that Y in (24) is log-normal.

$$\max E\int_0^\infty U(c,t)dt = E\int_0^\infty e^{-rt}\frac{c^\gamma}{\gamma}dt\,. \tag{25}$$

In market clearing equilibrium (see Lucas (1976) for stochastic dynamic optimization), the stock price at time *t* is:

$$S(t) = \frac{E_t\{\int_t^\infty U_c(\delta(s),s)\delta_s ds}{U_c(\delta(t),t)} = E_t\int_t^\infty \frac{U_c(\delta(s),s)}{U_c(\delta(t),t)}\delta_s ds\,. \tag{26}$$

The equation may be rewritten as:

$$S(t)\delta(t)^{r-1} = \int_t^\infty E_t e^{-r(s-t)}\delta(s)^{r-1}\delta(s)ds\,. \tag{27}$$

By substituting (24) into (27), the equilibrium stock price evolution is:

$$S(t) = S(s)\exp[(\alpha-\sigma^2/2-\lambda k)(s-t)+\sigma(B(t)-B(s))]\exp\sum_{j=N(s)+1}^{N(t)}Y_j\,. \tag{28}$$

This endogenizes Merton's stock price evolution with a mixture of geometric Brownian motion and the jump process.

Similarly, if a contingent claim pays $X = X(S(T)) = X(\delta(T))$ at maturity *T*, then (27) becomes:

$$S^X(t)\delta(t)^{r-1} = E_t[e^{-r(T-t)}\delta(T)^{r-1}X(\delta(T))]. \tag{29}$$

And the contingent claim price at any *t* is given by:

$$S^X(t) = E_t[e^{-r(T-t)} \frac{\delta(T)^{r-1}}{\delta(t)^{r-1}} X(\delta(T))] \quad (30)$$

where $\frac{\delta(T)^{r-1}}{\delta(t)^{r-1}}$ is the marginal rate of substitution.

Naik and Lee (1990) derived a call option with maturity T and exercise price K by using (30):

$$S^h(t) = \sum_{n=0}^{\infty} p(n) W(S(t), \pi_n, r_n, \sigma_n). \quad (31)$$

where $p(n) = \frac{e^{-r(T-t)}(\lambda(T-t))^n}{n!}$

and

$W(S(t), \pi_n, r_n, \sigma_n) = S(t)e^{-\pi_n(T-t)} N(d_{1,n}) - Ke^{-r_n(T-t)} N(d_{2,n})$

with

$r_n = r + (1-\gamma)\{\alpha + \mu_y \frac{n}{T-t} - \lambda k\} + \frac{1}{2}\{\sigma^2 + \sigma_y^2 \frac{n}{T-t}\}(\gamma-1)(2-\gamma)$,

$\pi_n = r - \gamma\{\alpha + \mu_y \frac{n}{T-t} - \lambda k\} - \frac{1}{2}\{\sigma^2 + \sigma_y^2 \frac{n}{T-t}\}\gamma(\gamma-1)$,

$\sigma_n^2 = \sigma^2 + \sigma_y^2 \frac{n}{T-t}$,

$d_{1,n} = \frac{\ln(S(t)/K) + (r_n - \pi_n + (1/2)\sigma_n^2)(T-t)}{\sigma_n \sqrt{T-t}}$,

$d_{2,n} = \frac{\ln(S(t)/K) + (r_n - \pi_n - (1/2)\sigma_n^2)(T-t)}{\sigma_n \sqrt{T-t}}$

The formula shows that Naik and Lee (1990) used a particular utility function with parameter γ and the original probability parameters α and λ. They did not use equivalent martingale measure. Merton (1976), on the other hand, used an equivalent martingale measure for the geometric Brownian motion portion of the mixture.

On the other hand, Davis (1994) proposed another utility maximizing method. An agent maximizes utility attained at maturity T by choosing a portfolio φ with initial capital x:

$$U^*(x) = \sup_{\varphi} E[U(V_{\varphi,x}(T))] = E[U(V_{\varphi^*,x}(T))]\,. \tag{32}$$

where φ^* is the optimal choice.

Now, she has the option of buying a contingent claim at price of p by spending an amount δ in addition to constructing a portfolio, and maximizes utility:

$$W(\delta, x, p) = \sup_{\varphi,\delta} E[U(V_{\varphi,x-\delta}(T) + \frac{\delta}{p} X)\,. \tag{33}$$

When the agent maximizes utility, she is indifferent between buying the option and not buying, i.e. $\frac{\partial W}{\partial \delta}(\delta, p, x)|_{\delta=0} = 0$. The restriction yields the price

$$p = E[\frac{U^{'}(V_{\varphi^*,x}(T))}{U^{*'}(x)} X] \tag{34}$$

where $\frac{U^{'}(V_{\varphi^*,x}(T))}{U^{*'}(x)}$ is the marginal rate of substitution.

The equation (30) and (34) are very similar. It is noted that both models are based on a self-financing portfolio strategy.

Variance Minimizing Hedging

We hedge non-attainable contingent claims as close as we can. Typically, the following optimization problem is considered.

Given a contingent claim X, minimize the quantity $E(X^{*} - V_{T}^{*}(\phi))^{2}$

over all self-financing trading strategies where superscript * means discounted values.

Notice that the quadratic terminal risk is simply the expected quadratic cost of revising the terminal portfolio in order to replicate a given claim. The optimal strategy is sought under the original probability measure. This problem in a discrete period model is examined by Schweizer (1992). Given a contingent claim X which settles at time T, we let $c(X)$ and $\phi(X)$ be the solution of the problem, i.e. initial capital and self-financing portfolio for the variance minimizing hedging. Schweizer (1995) showed that under mild assumptions, $c(X)$ and $\phi(X)$ can be found.

Esscher Transform

We already saw that there are many equivalent martingale measures in an incomplete market. In that circumstance, which martingale measure would we choose? The Esscher transform of a stock price process induces an equivalent probability measure. The particular Esscher parameter is determined *uniquely* so that the discounted price of the stock is a martingale under the corresponding probability measure. The Esscher transform is an efficient technique for valuing contingent claims if the logarithms of the stock prices are governed by certain stochastic processes with stationary and independent increments (Gerber and Shiu 1994).

Let $f(x)$ be a probability density function. Its moment generating function, possibly infinite, is $M(h) = \int e^{hx} f(x)dx$. The Esscher transform is defined by the new probability density for fixed h with. $M(h) < \infty$: $f(x;h) = \dfrac{e^{hx} f(x)}{M(h)}$. The new density function is parameterized by h such that $M(h) < \infty$.

There is a stochastic process $\{X(t)\}$ with stationary and independent increments such that $S(t) = S(0)e^{X(t)}$. Let $f(x,t)$ be a probability density function of a random variable $X(t)$. Its moment generating function is $M(h,t) = E(e^{hX(t)}) = \int e^{hX} f(x,t)dx$. Then, we can define a new density function $f(x,t;h) = \dfrac{e^{hx} f(x,t)}{M(h,t)}$ and it is the Esscher transform of the original density function. Each value of h generates an equivalent probability measure. The moment generating function $E^h(e^{zX(t)})$ of $X(t)$ when the density function corresponds to h is:

$$E^h(e^{zX(t)}) = M(z,t;h) = \int e^{zx} f(x,t;h)dx = \frac{M(z+h,t)}{M(h,t)}. \tag{33}$$

We like to find a particular h^* (or a particular equivalent martingale measure) such that $S(0) = E^{h^*}(e^{-Rt}S(t)) = E^{h^*}(e^{-Rt}S(0)e^{X(t)})$ or $e^{Rt} = E^{h^*}(e^{X(t)}) = M(1,t;h^*)$. Gerber and Shiu (1994) proposed the pricing of a contingent claim X at maturity T of the formula:

$$\pi(t) = e^{-R(T-t)} E^{h^*}(X \mid F_t) \tag{34}$$

where E^{h^*} is the expectation with respect to h^* equivalent martingale measure and F_t is information available at time t.

Gerber and Shiu (1994) derived the option price for the geometric Brownian motion (Black and Scholes 1973), and for the jump process with fixed size (Cox, Ross, and Rubinstein 1979) respectively. Their pricing by Esscher measure coincide the original formulas in these cases. It is not surprising or does not reveal any virtue of Esscher transform. These markets are complete and so there is a unique equivalent martingale measure for each market. Thus, Esscher measure equals the unique equivalent martingale measure in each case. The following example shows how convenient Essher transform is when there are many equivalent martingales.

Compound Poisson Process with Jump Size with Exponential Parameter δ

The stochastic process is described by

$X(t) = ct + \sum_{j=1}^{N(t)} U_j$ where $\{U_j\}$ are i.i.d. exponential with mean δ.

Then, the expected discounted stock price is:

$$Ee^{-rt}S(t) = S(0)e^{-rt}Ee^{X(t)} = S(0)e^{-rt}e^{(c-\lambda+\frac{\delta}{\delta-1}\lambda)t} = S(0)e^{(-r+c-\lambda+\frac{\delta}{\delta-1}\lambda)t}$$

(Note: The moment generating function of exponential is: $M_U(h) = (\frac{\delta}{\delta-h})$.)

Let us change the parameter values λ and δ such that $Ee^{-Rt}S(t) = S(0)$, i.e $-r + c - \lambda + \dfrac{\delta}{\delta - 1}\lambda = 0$. There is no unique λ and δ satisfying the equation. Thus, the Poisson process with exponential jump size does not have a unique martingale.

We now use the Esscher transform to find an equivalent martingale measure uniquely. The moment generating function of $X(t)$ is

$$M_{X(t)}(h) = M(h,t) = Ee^{hX(t)} = e^{-\lambda t}e^{\lambda t M_U(h)}e^{hct} = e^{(-\lambda + \lambda\frac{\delta}{\delta - h} + hc)t}. \tag{35}$$

The moment generating function of a new distribution parameterized by h is

$$M(z,t;h) = \frac{M(z+h,t)}{M(h,t)} = \frac{e^{[-\lambda + \lambda\frac{\delta}{\delta - z - h} + c(h+z)]t}}{e^{(-\lambda + \lambda\frac{\delta}{\delta - h} + ch)t}} = e^{(-\lambda\frac{\delta}{\delta - h} + \lambda\frac{\delta}{\delta - h}\frac{\delta - h}{\delta - z - h} + cz)t}. \tag{36}$$

Then, there is a unique h^* that satisfies $e^{Rt} = [M(1,t;h^*)]$ or $R = -\lambda\dfrac{\delta}{\delta - h^*} + \lambda\dfrac{\delta}{\delta - h^*}\dfrac{\delta - h^*}{\delta - h^* - 1} + c$. Thus, the Esscher measure makes $X(t)$ have new parameters $\lambda^* = \lambda\dfrac{\delta}{\delta - h^*}$ and $\delta^* = \delta - h^*$.

One question with the Esscher transform is that it mechanically chooses an equivalent martingale measure uniquely without having any economic rationale.

Gerber and Shiu (1994) argued that the choice of the Esscher measure may be justified by a utility maximizing argument. However, it is not clear that this is valid for the following reason.[8]

4. The Valuation of the Various methods for One Period Model

Suppose the stock price evolution has two components, an expected regular move and a rare abrupt move. The current stock price is 1. At the end of one period, the price can be $u = 1.1$ with probability of $p_1 = 0.59$, $d = 0.9$ with probability of $p_2 = 0.39$. However, the price can abruptly jump to j with probability of $p_3 = 0.02$. There is a call option with exercise price of 1. Let $\{a_1, a_2, a_3\}$ be a contingent claim payoff. Let the risk-free interest rate be $1 + r = 1.04$. If there is no jump, then we can find a unique martingale measure $p^* = \frac{1+r-d}{u-d} = 0.7$. Below we calculate the price for this example by each of method described in the previous sections.

Merton's Formula

Conditional on no jump, the stock price moves upward or downward. We use a unique equivalent martingale measure p^* to get the Black-Scholes's price.

[8] Utility maximization uses the original probability measure while Esscher transform an equivalent martingale measure. Therefore, it is not clear that Gerber and Shiu's (1994) argument is valid that Esscher transform can be justified by utility maximization.

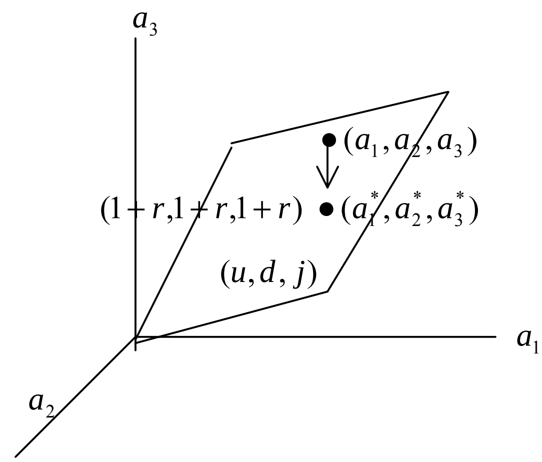
Conditional on jump, the stock price has only one value. The option price is $[\frac{1}{1+r}\{a_1 p^* + a_2(1-p^*)\}](1-p_3)+[\frac{1}{1+r}a_3]p_3$.

Variance Minimizing Hedging

Let α^* and β^* be the solution of

$$\min_{\alpha,\beta}[\{\alpha(1+r)+\beta u-a_1\}^2 p_1 \times +\{\alpha(1+r)+\beta d-a_2\}^2 p_2+\{\alpha(1+r)+\beta j-a_3\}^2 p_3]$$

Let (a_1^*, a_2^*, a_3^*) be the attainable contingent claim such that $\alpha^*(1+r)+\beta^* u = a_1^*$, $\alpha^*(1+r)+\beta^* d = a_2^*$, and $\alpha^*(1+r)+\beta^* j = a_3^*$. Then, the portfolio of α^* and β^* at the beginning evolves as close as possibly to the contingent claim (a_1, a_2, a_3) and minimizes risk. The contingent claim value under variance minimizing hedging is $\alpha^* + \beta^*$.



Under variance minimizing hedging, we value the attainable contingent claim (a_1^*, a_2^*, a_3^*), not the actual non-attainable contingent claim (a_1, a_2, a_3), and we need an equivalent martingale measure to use the risk neutral valuation as in Proposition 1. However, there are many sets of $\{ p_1^*, p_2^*, p_3^* \}$ that satisfies $1 + r = up_1^* + dp_2^* + jp_3^*$ and $1 = p_1^* + p_2^* + p_3^*$. Recall the earlier statement that an attainable contingent claim, even in the incomplete market, can be valued and the risk neural valuation by any martingale is the same. We prove this statement for this simple model.

Lemma 2: Let (a_1^*, a_2^*, a_3^*) be the attainable contingent claims under variance minimizing hedging and α^* and β^* be the portfolio. Then,

$(1+r)(\alpha^* + \beta^*) = (a_1^* p_1^* + a_2^* p_2^* + a_3^* p_3^*)$ for any (p_1^* , p_2^* , p_3^*) such that $1+r = up_1^* + dp_2^* + jp_3^*$ and $1 = p_1^* + p_2^* + p_3^*$.

Proof: Suppose there is a (p_1^*, p_2^*, p_3^*) such that $1+r = up_1^* + dp_2^* + jp_3^*$ and $1 = p_1^* + p_2^* + p_3^*$. The attainable contingent claim (a_1^*, a_2^*, a_3^*) is attained by the portfolio of α^* and β^* , i.e. $\alpha^*(1+r) + \beta^* u = a_1^*$, $\alpha^*(1+r) + \beta^* d = a_2^*$, $\alpha^*(1+r) + \beta^* j = a_3^*$. Then,

$(1+r)(\alpha^* + \beta^*) = (1+r)(p_1^* + p_2^* + p_3^*)\alpha^* + (up_1^* + dp_2^* + jp_3^*)\beta^*$ since

$1+r = up_1^* + dp_2^* + jp_3^*$ and $1 = p_1^* + p_2^* + p_3^*$.

$= \{\alpha^*(1+r) + \beta^* u\} p_1^* + \{\alpha^*(1+r) + \beta^* d\} p_2^* + \{\alpha^*(1+r) + \beta^* j\} p_3^*$

$= (a_1^* p_1^* + a_2^* p_2^* + a_3^* p_3^*)$.

Esscher Transform

The moment generating function is:

$$M(h) = e^{h \log u} p_1 + e^{h \log d} p_2 + e^{h \log j} p_3 = u^h p_1 + d^h p_2 + j^h p_3 .$$

Thus, the moment generating function of the Esscher-transformed random variable is:

$$M(z;h) = e^{z\log u}\frac{u^h p_1}{M(h)} + e^{z\log d}\frac{d^h p_2}{M(h)} + e^{z\log j}\frac{j^h p_3}{M(h)}.$$

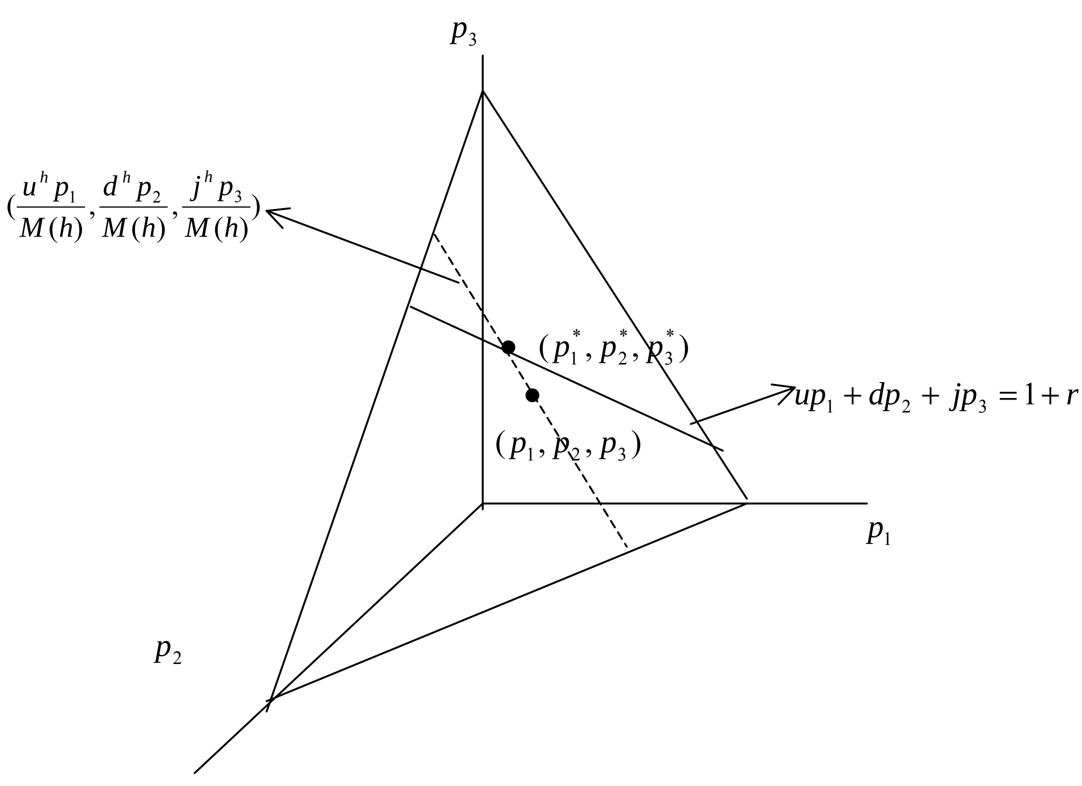
The parameter h^* that satisfies $1+r = M(1;h^*)$ yields the Esscher measure.

There is a unique solution h^* that satisfies:

$$M(1;h^*) = u\frac{u^{h^*} p_1}{M(h^*)} + d\frac{d^{h^*} p_2}{M(h^*)} + j\frac{j^{h^*} p_3}{M(h^*)} = 1+r.$$

Then, we let $\frac{u^{h^*} p_1}{M(h^*)} = p_1^*$, $\frac{d^{h^*} p_2}{M(h^*)} = p_2^*$, and $\frac{j^{h^*} p_3}{M(h^*)} = p_3^*$, and this

(p_1^*, p_2^*, p_3^*) is the Esscher measure.



The Esscher transform price is defined by $(1/1+r)(a_1 p_1^* + a_2 p_2^* + a_3 p_3^*)$. We can compare the Esscher transform price and the variance minimizing hedging price as follows. Note that the Esscher measure (p_1^* , p_2^* , p_3^*) satisfies $1+r = up_1^* + dp_2^* + jp_3^*$. By Lemma 2, the variance minimizing hedging price $\alpha^* + \beta^*$ is the expected discounted price of $(1/1+r)(a_1^* p_1^* + a_2^* p_2^* + a_3^* p_3^*)$ where the Esscher measure (p_1^*, p_2^*, p_3^*) is used. Thus, the variance minimizing hedging price and the Esscher transform price are $(1/1+r)(a_1^* p_1^* + a_2^* p_2^* + a_3^* p_3^*)$ and $(1/1+r)(a_1 p_1^* + a_2 p_2^* + a_3 p_3^*)$ respectively. Both of them use Esscher measure but a difference is that the minimizing variance price takes expectation of the attainable contingent claim that is closest to the actual non-attainable contingent claim, but the Esscher transform price takes expectation of the actual non-attainable contingent claim. Also the relation between the two prices is:

$$(1/1+r)(a_1 p_1^* + a_2 p_2^* + a_3 p_3^*)$$

$$=(1/1+r)(a_1^* p_1^* + a_2^* p_2^* + a_3^* p_3^*) + (1/1+r)[(a_1 - a_1^*) p_1^* + (a_2 - a_2^*) p_2^* + (a_3 - a_3^*) p_3^*].$$

That is, the Esscher transform price is obtained by compensating the variance minimizing hedging price by as much as the expected difference between the two claims under the Esscher martingale probability.

New Formula (Youn 2003)

Though Merton (1976) argued that his formula hedges risk conditional on jump, we do not know how much we hedge contingent claim risk with Merton's price because the price itself does not tell how to construct a replicating portfolio. Following Merton's idea, we suggest a new price formula. One is risk-neutral about jump risk because its probability is very small. Thus, she can try to hedge the contingent risk associated with jump separately.[9] Thus, we ignore the jump and take a portfolio of α^* and β^* to obtain a unique equivalent martingale p^* as if we are in the complete market. If there is no jump, the portfolio completely hedges the contingent claim perfectly. But we are exposed to the risk that the portfolio value is different from the contingent claim value when jump occurs. Since one is risk-neutral about the jump risk, she is compensated only by the difference between the portfolio value and the contingent claim value. Hence, the new proposed contingent claim value is defined by:

[9] If jump means natural disaster such as fire, we buy insurance and hedge the risk separately.

$$\frac{1}{1+r}\{a_1 p^* + a_2(1-p^*)\} - \frac{1}{1+r}(\alpha^*(1+r) + \beta^* j - a_3)p_3$$

where $\frac{1}{1+r}\{a_1 p^* + a_2(1-p^*)\} = (\alpha^* + \beta^*)$.

The variance minimizing hedging price tries to minimize risk but the chosen replicating portfolio may end with a value different from the contingent claims in every case. On the other hand, this new method hedges risk completely most of time, and compensates the differences that occur in rare cases.

The next table shows how the option value varies with each method. We see how each option value varies for three different jump sizes. As we indicated, the variance minimizing hedging price and Esscher transform price are close for each jump size. Our new method shows that as jump size is larger, the option value becomes higher because more gain is expected. It is noted that Merton's price is sizably different from the other prices.

J	Variance Minimizing Hedging	h*	Esscher	Youn	Merton
1.5	0.072454344	0.75	0.072432	0.071154	0.0712
2	0.075961538	0	0.075962	0.075962	0.0852
2.5	0.077037064	-0.3	0.077738	0.080769	0.0948

This approach is from the point of view of the contract holder. The price need not be fair to the writer from the point of view of arbitrage theory

Reference

Black, F and M. Scholes (1973), “The pricing of options and corporate liabilities”, Journal of Political Economy, 81, 637-654

Cox, J.C., S.A. Ross and M. Rubinstein (1979), “Option pricing: A simplified approach”, Journal of Financial Economics, 7, 229-268

Davis, M.H.A. (1997), “Option pricing in incomplete markets”, Dempster, M.A.H. and S.R. Pliska, editors, *Mathematics of derivative securities*, 216-226, Cambridge University Press

Gerber, H.U. and E.S.W. Shiu (1994), “Option pricing by Esscher transform, Transactions of the Society of Actuaries, 46, 99-140

Harrison., J.M. and S.R. Pliska (1981), “Martingale and stochastic integrals in the theory of continuous trading”, Stochastic Process Application, 11, 215-260

Lucas, R.E. (1978), “Asset prices in an exchange economy”, Econometrica, 46, 1429-1445

Merton, R.C. (1976), “Option pricing when underlying stock returns are discontinuous.” Journal of Financial Economics, 3, 125-144

Naik, V. and M. Lee (1990), “General equilibrium pricing of options on the market portfolio with discontinuous returns”, The Review of Financial Studies, 3, 493-521

Schweiser, M. (1992), “Mean variance hedging for general claim”, *Ann. Appl. Probab.* 2(1), 171-179

Schweiser, M. (1995), “On the minimal martingale measure and the Follmer-Schweiser decomposition” , *Stochast. Anal. Appl.* 13, 573-599

Von Neumann, J. and O. Morgenstern (1944), “Theory of games and economic behavior”, Princeton University