The Theory of Option Pricing in a Discrete Setting

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1. Introduction.

This paper will examine the theory of option pricing restricted to a discrete setting. There are two basic types of options. A *call option* is a contract which gives the holder the right to *purchase* an underlying asset at or before a specified date for a specified price. A *put option* is a contract which gives the holder the right to *sell* an underlying asset at or before a specified date for a specified price. The specified date is known as the *expiration date* or *expiry*, and the specified price is called the *exercise price* or *strike price*.

Options originated in Holland during the seventeenth century with trading on the purchase and sale of tulip bulbs. The first options traded in the United States were in agricultural commodities a century later. Until 1973, trading was sporadic and not regulated. The Chicago Board Options Exchange commenced the organized exchange of options on stocks. Today there are numerous types of options, including Asians (options on the average), barrier options (options which come into existence or become worthless if the asset reaches a prescribed value), binary options (cash or nothing calls which are "bets" on the asset price), lookback options (where the price depends on the maximum or minimum asset price), and even options on options (called compound options). Further, these options can be traded on a multitude of underlying assets, e.g. foreign currencies, mutual funds, stocks, bonds, and even agricultural products or metals.

In 1973, Fischer Black and Myron Scholes published "The Pricing of Options and Corporate Liabilities", a paper which revolutionized the theory of option pricing. Black and Scholes showed that there is a rational price for European options regardless of the opinions of the option traders. This result was surprising because it countered a common misconception that traders' predictions on the change in asset prices affected option prices. Additionally, this paper was revolutionary because Black and Scholes proved that the drift of the asset price has no effect on the value of the underlying asset. Using arbitrage considerations, they came up with a closed form solution for the price of a European call option. With this no arbitrage assumption they found that the value $V(t,S)$ of the call option at time t and with underlying asset S satisfies the final value problem

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} - rV = 0, \qquad V(T,S) = (S_T - k)^+$$

where r is the risk-free interest rate, and $V(T,S) = X(S) = (S_T - k)^+$ is the payoff function for the European call option. This is known as the Black-Scholes partial differential equation.

We choose to restrict this paper to the discrete setting for several reasons. First, two step binomial tree models illustrate many of the same concepts as more complicated models, yet they are much simpler to compute. It is unrealistic to only allow two possible outcomes for a change in asset price over the life of the asset; however, the assumption that the price movements are binomial in a short period of time Δt is reasonable. Relying on this assumption, Cox, Ross and Rubinstein first proposed the discrete time model for option valuation in 1979. The limiting case of this binomial model is the Black-Scholes model. Previously, depending on the payoff function, the Black-Scholes formula was obtained through more difficult numerical methods. Another reason we choose to examine the discrete setting is that option valuation is discretized in practice. Although with recent technology it has become possible to price options at much smaller time increments, pricing is still discrete in nature. The Black-Scholes model approximates the discrete practice of option pricing by letting the time increments diminish to zero.

The second section of this paper will give brief background definitions and theorems from probability theory and analysis. Most are directly applied in the paper; others are useful when reading related works. Here we also introduce a single factor binomial tree model in the context of a geometric random walk.

The third section introduces a finite setting for a $K+1$ dimensional stochastic process of underlying assets. Several examples of processes in this setting are developed here. With these examples, we illustrate concepts presented by Harrison and Pliska. More specifically, we will construct a self-financing strategy and find an equivalent martingale measure Q such that a discount process is a martingale under Q. With this measure Q we will illustrate a consistent price system. These concepts and others are presented theoretically in the paper of Harrison and Pliska and we attempt to make the ideas concrete, usually in reference to the single factor binomial tree model.

In the last section we arrive at the main result of this paper. Starting with the well known fact that there is no advantage to early exercise on an American call option without dividends, Thomann and Waymire conjectured that this would be the case for any convex payoff function with $V(0)=0$. In the fourth section we verify this conjecture for the discrete models described in Section 3. Various illustrative examples are also included.