A Note on the Distribution of Integrals of Geometric Brownian Motion*

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Abstract

The purpose of this note is to identify an interesting and surprising duality between the equations governing the probability distribution and expected value functional of the stochastic process defined by $A_t := \int_0^t \exp\{Z_s\} ds, t \geq 0$, where $\{Z_s : s \geq 0\}$ is a one-dimensional Brownian motion with drift coefficient μ and diffusion coefficient σ^2. In particular, both expected values of the form $v(t,x) := Ef(x + A_t)$, f homogeneous, as well as the probability density $a(t,y)dy := P(A_t \in dy)$ are shown to be governed by a pair of linear parabolic partial differential equations. Although the equations are not the backward/forward adjoint pairs one would naturally have in the general theory of Markov processes, unifying and remarkably simple derivations of these equations are provided.

1 Introduction

The vital role played by Kolmogorov's backward and forward equations for Markov processes is well-known from points of view of both general theory and

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computation. Of course these equations are intimately linked to the Markov structure as reflected in the semi-group property. Thus the appearance of a pair of linear parabolic (diffusion) partial differential equations in connection with the highly non-Markovian evolution of integrals of geometric Brownian motion seems quite noteworthy.

The stochastic process

$$A_t := \int_0^t \exp(Z_s) ds, \tag{1}$$

where $\{Z_s : s \geq 0\}$ is a Brownian motion with drift coefficient μ and diffusion coefficient $\sigma^2 > 0$, arises naturally in a diverse contexts involving geometric Brownian motion; eg. financial mathematics, spin-glass and disordered systems, statistical turbulence, to name a few. The purpose of this note is to identify an interesting and surprising duality between the equations governing the probability distribution and expected value functional of the stochastic process defined by (1). In particular, the expected values of the form

$$v(t, x) := Ef(x + A_t), \tag{2}$$

f a homogeneous function, as well as the probability density

$$a(t, y)dy := P(A_t \in dy), \tag{3}$$

will be shown to be governed by the following pair of linear parabolic (diffusion) partial differential equations

$$\frac{\partial v}{\partial t} = (1 + (\frac{1}{2} - \theta)\sigma^2 x - \mu x)\frac{\partial v}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 v}{\partial x^2} + (\mu\theta + \frac{1}{2}\sigma^2\theta^2)v \tag{4}$$

and

$$\frac{\partial a}{\partial t} = \frac{\partial^2}{\partial y^2}\{\frac{1}{2}\sigma^2 y^2 a\} - \frac{\partial}{\partial y}\{((\frac{1}{2}\sigma^2 + \mu)y + 1)a\}, t > 0, \quad a(0, y) = \delta_0(y). \quad (5)$$

Of course these equations are not the backward and forward adjoint pairs one would naturally have for the general theory of Markov processes, however unifying and remarkably simple derivations of these equations are forthcoming in this note.

Some related alternative approaches to formulae and equations governing (2) and (3) can be found in Yor (1992), Monthus and Comptet (1994), and Rogers and Shi (1995). Yor (1992) develops an identity from Bougerol (1983) and exploits connections between A_t and subordinated Brownian motion and Bessel processes in his derivation of a formula for the density of A_t. Monthus and Comtet (1994) give a non-rigorous evaluation of moment integrals to determine a recurrence relation for a formal power series representation of the moment generating function. This in turn leads to an equation for the density of A_t. Finally, Rogers and Shi (1995) exploit a particular scaling property to obtain a martingale whose drift term provides the appropriate pde governing the expected value in the case of *call function* $f(x) = x^+$. The essential feature of the call function exploited by Rogers and Shi (1995) is the implied (degree one) *homogeneity*. Namely the approach of Rogers and Shi (1995) applies to functions f such that for each real number x

$$f(\lambda x) = \lambda f(x), \lambda > 0. \quad (6)$$

The methods of the present note also permit an extension to *degree θ homogeneity* of the form

$$f(\lambda x) = \lambda^{\theta} f(x), \lambda > 0, \tag{7}$$

for some real parameter θ. While it is not our intent to develop the implications for numerical applications in this brief note, the utility of such partial differential equations for numerical calculations is nicely illustrated in Rogers and Shi (1995).

The organization is as follows. In the next section we provide the key lemma from which the equations are derived. The expected value equation and the equation for the probability density are then provided as rather immediate consequences.

2 Preliminaries and Key Lemma

Throughout we let $\{Z_t : t \geq 0\}$ be one-dimensional Brownian motion starting at 0 with drift μ and diffusion coefficient $\sigma^2 > 0$.

Remark The reader may take note that the calculations to follow extend to time-dependent drift and diffusion coefficients. The main property required of the diffusion $\{Z_t : t \geq 0\}$ is that for each $z \in \mathbf{R}$, the process $\{z + Z_t : t \geq 0\}$ starting from z has the same transition probabilities as the process $\{Z_t : t \geq 0\}$ starting from 0. Since the notation for temporally nonhomogeneous diffusions tends to complicate the underlying simplicity of the ideas, we have elected to restrict the presentation to the case of constant coefficients.

Lemma 2.1 *Define* $X_t^{(x)} = e^{-Z_t}(x + A_t), t \geq 0,$ *where*

$$A_t = \int_0^t \exp\{Z_s\} ds, \quad t \geq 0.$$

Then the bivariate process $\{(X_t^{(x)}, Z_t) : t \geq 0\}$ *is a Markov process with drift vector*

$$\mathbf{m} = (1 + \frac{1}{2}\sigma^2 x - \mu x, \mu)$$

and (singular) diffusion matrix

$$\mathbf{D} = \begin{pmatrix} \sigma^2 x^2 & -\sigma^2 x \\ -\sigma^2 x & \sigma^2 \end{pmatrix}.$$

Proof. The Markov property is a straightforward verification and will be left to the reader; cf also Carmona, Petit, and Yor (1997). Similarly, a straightforward application of Ito's lemma will provide

$$d(X_t^{(x)}, Z_t)' = \mathbf{m}' dt + \begin{pmatrix} 0 & -\sigma x \\ 0 & \sigma \end{pmatrix} (dW_t^{(1)}, dW_t^{(2)})',$$

where $'$ denotes matrix transpose. □

Remark. It is worthwhile to mention that a somewhat more general result than that of the simple Lemma 2.1 required here was shown to be of basic importance in related applications by Carmona, Petit,and Yor (1997).

Let us denote the infinitesimal generator of the Markov process $\{(X_t^{(x)}, Z_t) : t \geq 0\}$ by

$$\mathcal{L}^{(x,z)} = (1 + \frac{1}{2}\sigma^2 x - \mu x)\frac{\partial}{\partial x} + \mu\frac{\partial}{\partial z} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2} + \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial z^2} - \sigma^2 x \frac{\partial^2}{\partial x \partial z}. \quad (8)$$

3 Expected Value Equation

Let

$$v(t,x) = \mathbf{E} f(x + A_t), t \geq 0, \quad Z_0 = 0. \tag{9}$$

To obtain an equation for v we first write

$$u(t,x,z) = \mathbf{E}_{x,z} h(X_t, Z_t), t \geq 0, \tag{10}$$

where $h(x,z) = f(xe^z)$. Also, we omit the superscript x where it would be redundantly expressed as a subscript in the expected value symbol. Then for homogeneous f of degree θ one has

$$u(t,x,z) = e^{\theta z} v(t,x), t \geq 0, \tag{11}$$

and therefore

$$\frac{\partial u}{\partial z} = \theta u. \tag{12}$$

Thus applying Kolmogorov backward equation (eg. see Bhattacharya and Waymire,p. 374, 1990) and using (12) gives

$$\begin{aligned}\frac{\partial u}{\partial t} &= \mathcal{L}^{(x,z)} u(t,x,z) \\ &= (1 + \frac{1}{2}\sigma^2 x - \mu x)\frac{\partial u}{\partial x} + \mu\theta u + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 u}{\partial x^2} + \frac{1}{2}\sigma^2\theta^2 u - \sigma^2\theta x \frac{\partial u}{\partial x}\end{aligned} \tag{13}$$

Thus it now follows that

$$\frac{\partial v}{\partial t} = (1 + (\frac{1}{2} - \theta)\sigma^2 x - \mu x)\frac{\partial v}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 v}{\partial x^2} + (\mu\theta + \frac{1}{2}\sigma^2\theta^2)v. \tag{14}$$

This is precisely the equation obtained by Rogers and Shi (1995) in the case of homogeneous initial data of degree $\theta = 1$.

4 Probability Density Equation

Let

$$\hat{a}(t,\xi) := \mathbf{E}\exp\{i\xi A_t\}. \tag{15}$$

Also define

$$\varphi(t,z,\xi) := \mathbf{E}\exp\{i\xi e^z A_t\} = \mathbf{E}_{0,z} h_\xi(X_t, Z_t), \tag{16}$$

where for each fixed $\xi \in \mathbf{R}$

$$h_\xi(x,z) := \exp(i\xi x e^z). \tag{17}$$

Define

$$u(t,x,z;\xi) = \mathbf{E}_{x,z} h_\xi(X_t, Z_t) = \mathbf{E}_{0,0}\exp(i\xi(x + A_t)e^z). \tag{18}$$

Then

$$\varphi(t,z;\xi) = u(t,0,z;\xi) \tag{19}$$

and

$$\frac{\partial u}{\partial x}|_{x=0} = i\xi e^z u(t,0,z;\xi) = i\xi e^z \varphi(t,z;\xi). \tag{20}$$

Thus, again applying Kolmogorov's backward equation to u one obtains

$$\begin{aligned}\frac{\partial u}{\partial t} &= \mathcal{L}^{(x,z)} u(t,x,z) \\ &= (1 + \frac{1}{2}\sigma^2 x - \mu x)\frac{\partial u}{\partial x} + \mu\frac{\partial u}{\partial z} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 u}{\partial x^2} + \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial z^2} - \sigma^2 x \frac{\partial u^2}{\partial x \partial z}\end{aligned} \tag{21}$$

and therefore in view of (19) and (20) we have taking $x = 0$

$$\frac{\partial \varphi}{\partial t} = \mathcal{L}^{(z)}\varphi + i\xi e^z \varphi, \quad \varphi(0,z,\xi) = 1, \tag{22}$$

where

$$\mathcal{L}^{(z)} = \mu\frac{\partial}{\partial z} + \frac{1}{2}\sigma^2\frac{\partial^2}{\partial z^2}. \tag{23}$$

Next observe that

$$\hat{a}(t,\xi) := \varphi(t,0,\xi) = \mathbf{E}\exp\{i\xi A_t\}. \tag{24}$$

Differentiations of (16) with respect to z followed by evaluation at $z = 0$ yields

$$\frac{\partial\varphi}{\partial z}|_{z=0} = \mathbf{E} i\xi e^{z}A_t e^{i\xi e^{z}A_t}|_{z=0} = \xi\frac{\partial\hat{a}}{\partial\xi}.$$

and

$$\frac{\partial^2\varphi}{\partial z^2}|_{z=0} = \mathbf{E}\{(i\xi e^{z}A_t)^2 e^{i\xi e^{z}A_t} + i\xi e^{z}A_t e^{i\xi e^{z}A_t}\}|_{z=0} = \xi^2\frac{\partial^2\hat{a}}{\partial\xi^2} + \xi\frac{\partial\hat{a}}{\partial\xi}.$$

Substituting into (22) we obtain

$$\frac{\partial\hat{a}}{\partial t} = \frac{1}{2}\sigma^2\xi^2\frac{\partial^2\hat{a}}{\partial\xi^2} + (\frac{1}{2}\sigma^2 + \mu)\xi\frac{\partial\hat{a}}{\partial\xi} + i\xi\hat{a}. \tag{25}$$

One may now obtain the desired equation for the density $a(t,y)$ by applicat ion of the Fourier inversion formula

$$a(t,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\xi y}\hat{a}(t,\xi)d\xi$$

to (25). Specifically, the distribution of A_t is absolutely continuous for each $t > 0$ with probability density $a(t,y)$ satisfying

$$\frac{\partial a}{\partial t} = \frac{\partial^2}{\partial y^2}\{\frac{1}{2}\sigma^2 y^2 a\} - \frac{\partial}{\partial y}\{((\frac{1}{2}\sigma^2 + \mu)y + 1)a\}, t > 0, \quad a(0,y) = \delta_0(y). \tag{26}$$

Remark. A very simple and effective alternative approach to obtain an equation for $\hat{a}(t,\xi)$ is to directly apply the Feynman-Kac formula to $\varphi(t,z,\xi)$ to obtain (22). The remainder of the derivation is then the same.

Notice that if one considers the naturally associated diffusion $\{X_t : t \geq 0\}$ defined by

$$dX = \gamma(X_t)dt + \eta(X_t)dW \tag{27}$$

where

$$\gamma(x) = 1 + (\frac{1}{2}\sigma^2 + \mu)x, \quad \eta(x) = \sigma x, \tag{28}$$

having transition probabilities $p(t, x, y)$, then as an immediate consequence one has that

$$a(t, y) = p(t, 0, y). \tag{29}$$

Thus while the auxiliary diffusion $\{X_t : t \geq 0\}$ *overdetermines* the probability density $a(t, y)$, it explicitly makes available a larger system of pde's; namely both the Kolmogorov forward and backward equations for $p(t, x, y)$.

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