The Fundamental Solution of the Linearized Navier Stokes Equations for Spinning Bodies in Three Spatial Dimensions - Time Dependent Case

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To Appear in Journal of Math Fluid Mech

Abstract

Explicit formulae for the fundamental solution of the linearized time dependent Navier Stokes equations in three spatial dimensions are obtained. The linear equations considered in this paper include those used to model rigid bodies that are translating and rotating at a constant velocity. Estimates extending those obtained by Solonnikov in [23] for the fundamental solution of the time dependent Stokes equations, corresponding to zero translational and angular velocity, are established. Existence and uniqueness of solutions of these linearized problems is obtained for a class of functions that includes the classical Lebesgue spaces $L^p(\mathbf{R}^3), 1 < p < \infty$. Finally, the asymptotic behavior and semigroup properties of the fundamental solution are established.

Key Words: Fundamental solution, Linearized Navier Stokes equations, spinning bodies, Oseen and Stokes flow, Exterior Problems.

Mathematical Subject Classification (2000): 31B99, 33C15, 35Q30, 76D07.

1 Introduction and statement of the main results

The determination of explicit formulae for the fundamental solution of the linearized fluid equations is useful in establishing basic properties of the nonlinear

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*Both authors partially supported by NSF 0073958. RBG also supported by ONR grant N00014-02-1-0116

Navier Stokes equations. For example, spatial decay properties for the velocity field of the steady exterior problem were obtained by Finn in [5], [7] and for the vorticity by Clark [3]. These estimates use the Oseen fundamental tensor in free space for the steady problem resulting in similar estimates for the solution of the nonlinear steady Navier Stokes equations. In particular, using these decay estimates, Guenther et al [12] show that the drag on a rigid body that is moving at a constant velocity can be determined from the far field behavior of the pressure. Consequently, it is natural to conjecture that knowledge of the decay properties of the fundamental solution for the linearized time dependent problem could also be used to obtain similar decay estimates for the solution of the nonlinear problem and for the evaluation of the hydrodynamic forces in the case of spinning bodies.

Insight into the solvability and properties of the solution of boundary value problems could also be gained from a knowledge of the fundamental solution of the free space problem as is well known from classical potential theory. (See e.g. [13].) Similarly, for numerical calculations, the knowledge of the fundamental solution can be used to develop boundary element methods applicable in very general domains (See e.g. Schatz et al [20]). At the same time, for the Stokes and Oseen equations, the drag on a rigid body can be determined by solving integral equations of the first kind which are also determined in terms of the fundamental solution of the linearized problem (Youngren et al, [27], [28], Schuster [21]). It is natural to speculate that with the knowledge of the fundamental solution in the time dependent case and for spinning bodies, a similar program could be carried out.

From the work of Oseen [19], and Odqvist [18] and Lichtenstein [15], explicit formulae for the fundamental solution of the linearized fluid flow equations are known for steady Oseen and Stokes flows. The purpose of this paper is to present the corresponding fundamental solution for the time dependent equations that model an incompressible fluid in the exterior of a rigid body that is translating and rotating at a constant velocity and rotation. Specifically, consider the linear problem

$$\frac{\partial \mathbf{v}}{\partial t} - (\mathbf{U} + \omega \wedge \mathbf{y}) \cdot \nabla \mathbf{v} + \omega \wedge \mathbf{v} = \Delta \mathbf{v} - \nabla \pi + \mathbf{F} \qquad \nabla \cdot \mathbf{v} = 0 \;\text{ for } t > 0, \quad (1.1)$$

with initial data

$$\mathbf{v}(\mathbf{y}, 0) = \mathbf{u}_0(\mathbf{y}).$$

Here, $\Delta = \sum_{j=1}^{3} \partial^2 / \partial y_j^2$ stands for the Laplace operator and $\mathbf{F}(\mathbf{y}, t)$ represents a forcing term. Let

$$\mathcal{L}(\mathbf{v}) = \frac{\partial \mathbf{v}}{\partial t} - (\mathbf{U} + \omega \wedge \mathbf{y}) \cdot \nabla \mathbf{v} + \omega \wedge \mathbf{v} - \Delta \mathbf{v} \equiv \frac{\partial \mathbf{v}}{\partial t} - \mathcal{L}_{\mathbf{y}}(\mathbf{v}).$$

One is interested in solving (1.1) for a large enough class of initial data and forces that includes $(\mathcal{S}(\mathbf{R}^3))^3$ where $\mathcal{S}(\mathbf{R}^3)$ is the usual Schwartz class of test functions. As in [23] the **Fundamental Tensor** is comprised of a 3×3 matrix of distributions $\mathbf{\Gamma}(\mathbf{y}, \mathbf{z}, t)$ and a three dimensional vector of distributions $\mathbf{Q}(\mathbf{y}, \mathbf{z}, t)$ such that for arbitrary $\mathbf{A} \in \mathbf{R}^3$ and $t \geq s$, the distributions $\mathbf{v}_{\mathbf{A}}(\mathbf{y}, t) = \mathbf{\Gamma}(\mathbf{y}, \mathbf{z}, t-s)\mathbf{A}$, $\pi_{\mathbf{A}}(\mathbf{y}, t) = \mathbf{Q}(\mathbf{y}, \mathbf{z}, t-s) \cdot \mathbf{A}$, satisfy for $t \geq s$, in the sense of distributions,

$$\frac{\partial \mathbf{v}_{\mathbf{A}}}{\partial t} - \mathcal{L}_{\mathbf{y}}(\mathbf{v}_{\mathbf{A}}) + \nabla \pi_{\mathbf{A}} = \delta_s(t) \delta_{\mathbf{z}}(\mathbf{y}) \mathbf{A}, \qquad \nabla \cdot (\mathbf{v}_{\mathbf{A}}) = 0.$$

Here, $\delta_s(t)$ and $\delta_{\mathbf{z}}(\mathbf{y})$ denote the point mass concentrated at $t = s$ and $\mathbf{y} = \mathbf{z}$, respectively.

The main result of this paper determines the matrix $\mathbf{\Gamma}$ in terms of

$$K(\mathbf{x}, t) = \frac{1}{(4\pi t)^{3/2}} \exp\left(-\frac{|\mathbf{x}|^2}{4t}\right),$$

the fundamental solution of the heat equation in $\mathbf{R}^3$, and the Kummer or confluent hypergeometric function, ${}_1F_1(1, 5/2, u)$. In the statement of this result, Ω denotes the 3×3 matrix such that

$$\omega \wedge \mathbf{x} = \Omega \mathbf{x},$$

whereas $\mathbf{x} \otimes \mathbf{y}$ denotes the tensor product of two vectors in $\mathbf{R}^3$ given by $(\mathbf{x} \otimes \mathbf{y})_{ij} = x_i y_j$.

Theorem 1.1 *Let* $\mathbf{z}(t) = e^{-t\Omega}\mathbf{z} - t\mathbf{U}$*. The fundamental tensor of the linear problem (1.1) is given by*

$$\begin{aligned} \mathbf{\Gamma}(\mathbf{y}, \mathbf{z}, t) \quad = \quad & K(\mathbf{y} - \mathbf{z}(t), t) \left\{ \left[I - \frac{(\mathbf{y} - \mathbf{z}(t)) \otimes (\mathbf{y} - \mathbf{z}(t))}{|\mathbf{y} - \mathbf{z}(t)|^2} \right] \right. \\ & \left. - \ {}_1F_1\left(1, \frac{5}{2}, \frac{|\mathbf{y} - \mathbf{z}(t)|^2}{4t}\right) \left[\frac{1}{3} I - \frac{(\mathbf{y} - \mathbf{z}(t)) \otimes (\mathbf{y} - \mathbf{z}(t))}{|\mathbf{y} - \mathbf{z}(t)|^2} \right] \right\} e^{-t\Omega} \end{aligned}$$

$$\mathbf{Q}(\mathbf{y}, \mathbf{z}, t) = \frac{-1}{4\pi} \nabla_{\mathbf{y}} \frac{1}{|\mathbf{y} - \mathbf{z}(t)|} \delta_0(t) \equiv \mathbf{Q}^*(\mathbf{y}, \mathbf{z}(t)) \delta_0(t).$$

From this explicit formula it is possible by direct inspection to extend the validity of some of the estimates obtained in [23] to rotating rigid bodies. For example one has that as $\mathbf{y} \to \infty$, and for $t > 0$ fixed,

$$|\mathbf{\Gamma}_{ij}(\mathbf{y}, \mathbf{z}, t)| \sim \frac{3}{4\pi} \frac{1}{|\mathbf{y}|^3}$$

with similar estimates for the time and spatial derivatives of $\mathbf{\Gamma}$. The precise statements are given in Propositions 4.1 and 4.3. In particular, it should be noted that if $\mathbf{u}$ a physically relevant solution of the Navier Stokes equation as defined by Finn in [6], [5], (see also Galdi [9]) then

$$\int_{\mathbf{R}^3} |\mathbf{\Gamma}(\mathbf{y}, \mathbf{z}, t-s)\mathbf{u}(\mathbf{z}, s)| d\mathbf{z} < \infty.$$

The uniqueness of solutions for the linearized problem (1.1) can be obtained using the fundamental tensor from Theorem 1.1 following standard methods.

Theorem 1.2 *Let* $T > 0$ *and assume that for some* $1 < p < \infty$, $0 < \alpha$, $\mathbf{u}_0(\mathbf{y}) \in (L^p(\mathbf{R}^3))^3$, $\mathbf{F}(\mathbf{y}, t) \in \mathcal{C}([0, T], (L^p(\mathbf{R}^3))^3) \cap \mathcal{C}^{1+\alpha}(\mathbf{R}^3 \times [0, T])$ *and* $\nabla \cdot \mathbf{u}_0 = 0$. *Then the unique solution of*

$$\frac{\partial \mathbf{v}}{\partial t} - (\mathbf{U} + \omega \wedge \mathbf{y}) \cdot \nabla \mathbf{v} + \omega \wedge \mathbf{v} = \Delta \mathbf{v} - \nabla \pi + \mathbf{F} \qquad \nabla \cdot \mathbf{v} = 0 \quad \text{for } t > 0,$$

with initial data $\mathbf{v}(0, \mathbf{y}) = \mathbf{u}_0(\mathbf{y})$ *and* $(\mathbf{v}, p) \in \mathcal{C}([0, T], (L^p(\mathbf{R}^3))^4)$ *is given by*

$$\mathbf{v}(\mathbf{y}, t) = \int_0^t \int_{\mathbf{R}^3} \mathbf{\Gamma}(\mathbf{y}, \mathbf{z}, t-s) \mathbf{F}(\mathbf{z}, s) d\mathbf{z} ds + \int_{\mathbf{R}^3} \mathbf{\Gamma}(\mathbf{y}, \mathbf{z}, t) \mathbf{u}_0(\mathbf{z}) d\mathbf{z},$$

$$p(\mathbf{y}, t) = \int_{\mathbf{R}^3} \mathbf{Q}^*(\mathbf{y}, \mathbf{z}(t)) \cdot \mathbf{F}(\mathbf{z}, t) d\mathbf{z}.$$

Semigroup properties of the tensor $\mathbf{\Gamma}(\mathbf{y}, \mathbf{z}, t)$ follow from this Theorem and the estimates obtained in Proposition 4.3. Denote by $\mathcal{L}^*$ the formal adjoint of $\mathcal{L}$ given by

$$\mathcal{L}^*(\mathbf{v}) = -\frac{\partial \mathbf{v}}{\partial s} + (\mathbf{U} + \omega \wedge \mathbf{z}) \cdot \nabla \mathbf{v} - \omega \wedge \mathbf{v} - \Delta \mathbf{v} = -\frac{\partial \mathbf{v}}{\partial s} - \mathcal{L}^*_{\mathbf{z}}(\mathbf{v})$$

where the spatial differentiation should be interpreted to be with respect to the components of $\mathbf{z}$.

Theorem 1.3 *The fundamental solution* $\mathbf{\Gamma}(\mathbf{y}, \mathbf{z}, t-s)$ *from Theorem 1.1 is unique. As a function of* $(\mathbf{y}, t)$ *it satisfies for* $(\mathbf{y}, t) \neq (\mathbf{z}, s)$, $\mathcal{L}(\mathbf{\Gamma A}) = 0$, $\nabla\cdot(\mathbf{\Gamma A}) = 0$ *for any* $\mathbf{A} \in \mathbf{R}^3$, *and as a function of* $(\mathbf{z}, s)$ *its transpose,* $\mathbf{\Gamma}'$, *satisfies the adjoint problem* $\mathcal{L}^*(\mathbf{\Gamma}'\mathbf{A}) = 0$, $\nabla_{\mathbf{z}} \cdot (\mathbf{\Gamma}'\mathbf{A}) = 0$. *Furthermore, for* $\mathbf{F} \in (\mathcal{S}(\mathbf{R}^3))^3$,

$$\lim_{(\mathbf{y},t)\to(\mathbf{y}^0,0^+)} \int_{\mathbf{R}^3} \mathbf{\Gamma}(\mathbf{y}, \mathbf{z}, t)\mathbf{F}(\mathbf{z})d\mathbf{z} = \mathbf{H}(\mathbf{y}^0)$$

where $\mathbf{H}(\mathbf{y})$ *is the projection of* $\mathbf{F}(\mathbf{y})$ *onto divergence free vector fields, and*

$$\int_{\mathbf{R}^3} \mathbf{\Gamma}(\mathbf{y}, \mathbf{z}', t-\tau)\mathbf{\Gamma}(\mathbf{z}', \mathbf{z}, \tau - s)d\mathbf{z}' = \mathbf{\Gamma}(\mathbf{y}, \mathbf{z}, t-s).$$

Remark 1 *In Theorem 1.2 the condition* $\nabla\cdot\mathbf{u}_0 = 0$ *is not essential. In general, if the initial data does not satisfy the incompressibility conditions, the velocity remains unchanged but the pressure at time* $t = 0$ *is given by*

$$p(\mathbf{y}, t) = \int_{\mathbf{R}^3} \mathbf{Q}^*(\mathbf{y}, \mathbf{z}(t)) \cdot (\mathbf{F}(\mathbf{z}, t) + \mathbf{u}_0(\mathbf{z})\ d\mathbf{z}.$$

The organization of this paper is as follows. In the next section, a formulation of the problem under consideration is given. In particular the relation with the Navier Stokes equations in the exterior of a translating and spinning rigid body is reviewed. This section also includes basic properties of the Kummer functions, a review of the use of the Riesz transforms in this context as well as setting up the notation used in the paper. In Section 3, the proof of the main theorem is given. Section 4 contains estimates for the derivatives of the fundamental solution tensor and the proof of the uniqueness of solutions and the semigroup property of the fundamental solution. Section 5 contains conclusion and remarks.

2 Problem Formulation

Consider the time dependent Navier Stokes equations in a system of coordinates with its origin at the center of mass of a rigid body translating with constant velocity $\tilde{\mathbf{U}}$ and rotating at constant angular velocity $|\tilde{\omega}|$ with respect to an axis $\tilde{\omega}/|\tilde{\omega}|$. Denote by $\mathcal{D}$ the region exterior to the rigid body and $\partial\mathcal{D}$ its boundary.

From the work of Weinberger in [25], [26] and Serre in [22], the Navier Stokes equations in this system of coordinates can be written in nondimensional form as

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} - (\mathbf{U} + \omega \wedge \mathbf{x})\cdot\nabla\mathbf{u} + \omega\wedge\mathbf{u} = \frac{1}{\mathcal{R}e}\Delta\mathbf{u} - \nabla p - \mathcal{Q}\mathbf{g}, \qquad \nabla\cdot\mathbf{u} = 0.$$

See the comprehensive work of Galdi [10] for a contemporary exposition of the use of these equations in sedimentation and other problems and the more recent [11] for the existence of physically reasonable, in the sense of Finn, solutions for slowly spinning bodies. In these equations, $\mathbf{u}$ is the non-dimensionalized fluid velocity in the described system of coordinates, $\mathcal{R}e$ is the Reynolds number, p is the pressure divided by the Euler number $\mathcal{E}u = P/|\tilde{\mathbf{U}}|\rho$, $\mathbf{g} =< 0, 0, \gamma >$, $\gamma = gL/\tilde{U}^2$ is the reciprocal of the Froude number, $\mathbf{U} = \tilde{\mathbf{U}}/\tilde{U}$, $\omega = (L/\tilde{U})\tilde{\omega}$ and L and $\tilde{U} = |\tilde{\mathbf{U}}|$ are characteristic rigid body lengths and body current speed, respectively, and P is a characteristic pressure. Finally, the matrix $\mathcal{Q}$ satisfies

$$\dot{\mathcal{Q}}\mathbf{x} = \mathcal{Q}(\omega\wedge\mathbf{x}).$$

Note that introducing Ω to be the antisymmetric matrix such that $\Omega\mathbf{x} = \omega\wedge\mathbf{x}$, one has that $\mathcal{Q}(t) = \exp(t\Omega)$.

The boundary conditions are that the velocity decays at infinity and that on the surface $\partial\mathcal{D}$ the velocity of the fluid equals the velocity of the solid;

$$\mathbf{u}(\mathbf{x}) = \mathbf{U} + \omega \wedge \mathbf{x}, \quad \text{for } \mathbf{x} \in \partial\mathcal{D}.$$

The linearized equations, similar to the Oseen equations, are given by

$$\frac{\partial \mathbf{u}}{\partial t} - (\mathbf{U} + \omega \wedge \mathbf{x}) \cdot \nabla \mathbf{u} + \omega \wedge \mathbf{u} = \frac{1}{\mathcal{R}e} \Delta \mathbf{u} - \nabla p - \mathcal{Q}\mathbf{g}, \qquad \nabla \cdot \mathbf{u} = 0.$$

Equation (1.1) serves as a model for this physical equation by taking $\mathbf{U}' = \mathcal{R}e\ \mathbf{U}, \omega' = \mathcal{R}e\ \omega$, scaling time as $t' = t/\mathcal{R}e$, and dropping the prime notation.

The determination of the fundamental solution for this linear problem is done in the next section using the Riesz transforms in $\mathbf{R}^3$. This formula involves the Kummer function, ${}_1F_1(1, 5/2, u)$. For completeness, a summary of basic properties of the Riesz transforms and Kummer functions needed for this paper is included here.

First recall that for $f \in L^p(\mathbf{R}^3)$, $p > 1$, the Riesz transforms $\mathcal{R}_j$, $j = 1, 2, 3$, are given by

$$\mathcal{R}_j(f)(\mathbf{y}) = \frac{1}{\pi^2} \int \frac{y_j - x_j}{|\mathbf{y} - \mathbf{x}|^4} f(\mathbf{x})\ d\mathbf{x}.$$

(See Stein [24] for example.) Defining the Fourier transform of a function in $\mathcal{S}(\mathbf{R}^3)$ by

$$\mathcal{F}(f)(\xi) = \frac{1}{(2\pi)^{3/2}} \int e^{-i\mathbf{x}\cdot\xi} f(\mathbf{x}) d\mathbf{x},$$

one has, for $f, g \in \mathcal{S}(\mathbf{R}^3)$,

$$\mathcal{F}(f * g)(\xi) = (2\pi)^{3/2} \mathcal{F}(f)(\xi) \mathcal{F}(g)(\xi).$$

Then,

$$\mathcal{F}(\mathcal{R}_j(f))(\xi) = -i\frac{\xi_j}{|\xi|}\mathcal{F}(f)(\xi).$$

In particular, let $\mathcal{P}$ denote the projection on divergence free vector fields. Then it follows immediately that in the sense of distributions,

$$\mathcal{P} = I + \mathcal{R} \tag{2.1}$$

where $\mathcal{R}$ is the matrix with ij^{th} entry given by $\mathcal{R}_i\mathcal{R}_j$.

With regards to the properties of the Kummer functions needed for this work, recall that

$${}_1F_1(a, c, u) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n}\frac{u^n}{n!}$$

where $(b)_n = \Gamma(b+n)/\Gamma(b)$ is the Pochhammer symbol. In particular, $(1)_n = n!$ so

$${}_1F_1(1, c, u) = \sum_{n=0}^{\infty} \frac{1}{(c)_n}u^n.$$

The basic differentiation property of the Kummer functions (see [16], pg 264) that is used in this paper is

$$\frac{d\ {}_1F_1(a, c, u)}{du} = \frac{a}{c}\ {}_1F_1(a+1, c+1, u). \tag{2.2}$$

Also, while many recursion formulae are available for these confluent hypergeometric functions, the one most useful for this paper is displayed as the third formula on page 268 of [16]:

$$c\ {}_1F_1(a, c, u) - (c-a)\ {}_1F_1(a, c+1, u) - a\ {}_1F_1(a+1, c+1, u) = 0 \tag{2.3}$$

Finally, the asymptotic behavior as $u \to \infty$ of the Kummer functions can be written as

$$e^{-u}\ {}_1F_1(1,c,u) \sim \Gamma(c)\frac{1}{u^{c-1}} \quad \text{as } u \to \infty. \tag{2.4}$$

(See [16] page 289.)

In the derivation of some of the results, the following property of the Kummer functions is useful.

Lemma 2.1 *Let* $c > 0$. *Then*

$$\frac{de^{-u}\ {}_1F_1(1,c,u)}{du} = e^{-u}\left(\frac{1-c}{c}\right)\ {}_1F_1(1,c+1,u).$$

Proof: The result follows using the noted properties (2.2) and (2.3) of the Kummer functions. Indeed,

$$\begin{aligned}\frac{de^{-u}\ {}_1F_1(1,c,u)}{du} &= e^{-u}\left[\frac{1}{c}\ {}_1F_1(2,c+1,u) - \ {}_1F_1(1,c,u)\right]\\ &= e^{-u}\frac{1}{c}\left[\ {}_1F_1(2,c+1,u) - c\ {}_1F_1(1,c,u)\right]\\ &= e^{-u}\frac{1}{c}(-1)(c-1)\ {}_1F_1(1,c+1,u).\blacksquare\end{aligned}$$

3 Proof of the Main Theorem

To obtain the fundamental solution of (1.1) consider first the linear operator

$$\tilde{\mathcal{L}}(W) = \Delta W + (\mathbf{U} + \omega \wedge \mathbf{y}) \cdot \nabla W - \omega \wedge W \tag{3.1}$$

The fundamental solution of this linear system of equations can be obtained in terms of the three dimensional heat kernel as indicated in the following Proposition. It is convenient to introduce the following notation;

$$\mathbf{z}(t) = e^{-t\Omega}\mathbf{z} - t\mathbf{U}. \tag{3.2}$$

Proposition 3.1 *Assume that* $\tilde{\mathbf{u}}_0(\mathbf{z}) \in \mathcal{S}(\mathbf{R}^3)$. *Define*

$$\tilde{\mathbf{\Gamma}}(\mathbf{y}, \mathbf{z}, t) = K(\mathbf{y} - \mathbf{z}(t), t)e^{-t\Omega}. \tag{3.3}$$

Then

$$\tilde{\mathbf{u}}(\mathbf{y}, \mathbf{z}, t) = \int_{\mathbf{R}^3} \tilde{\mathbf{\Gamma}}(\mathbf{y}, \mathbf{z}, t)\tilde{\mathbf{u}}_0(\mathbf{z}) \; d\mathbf{z}$$

satisfies the linear problem

$$\frac{\partial \tilde{\mathbf{u}}}{\partial t} = \tilde{\mathcal{L}}\tilde{\mathbf{u}} \quad \text{for } t > 0,$$

with initial data $\tilde{\mathbf{u}}(\mathbf{y}, 0) = \tilde{\mathbf{u}}_0(\mathbf{y})$.

Proof: The proposition is established by performing a sequence of changes of dependent and independent variables.

Let $\mathbf{u}^* = \exp(t\Omega)\tilde{\mathbf{u}}$. Then a simple calculation, shows that $\mathbf{u}^*$ satisfies

$$\frac{\partial \mathbf{u}^*}{\partial t} - (\mathbf{U} + \omega \wedge \mathbf{y}) \cdot \nabla \mathbf{u}^* = \Delta \mathbf{u}^*$$

Now, let $\mathbf{v}(\mathbf{y}, t) = \mathbf{u}^*(\mathbf{y} - \mathbf{U}t, t)$. Then in terms of this unknown the linear problem to be solved is

$$\frac{\partial \mathbf{v}}{\partial t} - (\omega \wedge \mathbf{y}) \cdot \nabla \mathbf{v} = \Delta \mathbf{v}. \tag{3.4}$$

From the results in DaPrato and Lunardi [4], see also Chen and Miyakawa [2], it follows that the solution of the initial value problem (3.4) with initial data $\tilde{\mathbf{u}}_0(\mathbf{y})$ is given by

$$\begin{aligned} \mathbf{v}(\mathbf{y}, t) &= \frac{1}{(4\pi t)^{3/2}} \int_{\mathbf{R}^3} e^{-|\mathbf{z}|^2/4t} \tilde{\mathbf{u}}_0(\exp(t\Omega)\mathbf{y} - \mathbf{z}) d\mathbf{z} \\ &= \frac{1}{(4\pi t)^{3/2}} \int_{\mathbf{R}^3} \exp\left(-\frac{|e^{t\Omega}\mathbf{y} - \mathbf{z}|^2}{4t}\right) \tilde{\mathbf{u}}_0(\mathbf{z}) d\mathbf{z}. \end{aligned}$$

Since $\tilde{\mathbf{u}}(\mathbf{y},t) = \exp(-t\Omega)\mathbf{v}(\mathbf{y}+t\mathbf{U},t)$, it follows that

$$\tilde{\mathbf{u}}(\mathbf{y},t) = \frac{1}{(4\pi t)^{3/2}} e^{-t\Omega} \int_{\mathbf{R}^3} \exp\left(-\frac{|e^{t\Omega}(\mathbf{y}+t\mathbf{U}) - \mathbf{z}|^2}{4t}\right) \tilde{\mathbf{u}}_0(\mathbf{z}) d\mathbf{z} \tag{3.5}$$

is the solution of

$$\frac{\partial \tilde{\mathbf{u}}}{\partial t} = \tilde{\mathcal{L}}\tilde{\mathbf{u}} \tag{3.6}$$

with initial data $\tilde{\mathbf{u}}_0$. The proposition follows by noting that $|e^{t\Omega}(\mathbf{y}+t\mathbf{U}) - \mathbf{z}| = |\mathbf{y} - (e^{-t\Omega}\mathbf{z} - t\mathbf{U})|$. ■

To obtain the fundamental solution of the linearized problem (1.1) taking into account the incompressibility condition, the calculations leading to Proposition 3.1 need to be adapted. Using standard arguments it is easy to see that the 3×3 matrix $\mathbf{\Gamma}(\mathbf{y},\mathbf{z},t)$ defined by

$$\mathbf{\Gamma}\mathbf{A} = \mathcal{P}(\tilde{\mathbf{\Gamma}}\mathbf{A})$$

and

$$\mathbf{Q}(\mathbf{y},\mathbf{z}) = -\frac{1}{4\pi}\nabla\frac{1}{|\mathbf{y}-\mathbf{z}|}\delta_0(t).$$

form the fundamental tensor for the linear equation (1.1). In particular, for $t > 0$,

$$\mathcal{L}(\mathbf{\Gamma}\mathbf{A}) = 0, \qquad \nabla\cdot(\mathbf{\Gamma}\mathbf{A}) = 0. \tag{3.7}$$

In preparation for obtaining the explicit formula for $\mathbf{\Gamma}$, note that from the relation between the projection $\mathcal{P}$ and the Riesz transforms noted in (2.1), it follows that $\mathbf{\Gamma}$ can be expressed in terms of the fundamental solution of the three dimensional heat equation. Indeed, let $\mathbf{A} \in \mathbf{R}^3$ be fixed. Then

$$\mathbf{\Gamma}(\mathbf{y},\mathbf{z},t)\mathbf{A} \quad = \quad \mathcal{P}\tilde{\mathbf{\Gamma}}(\mathbf{y},\mathbf{z},t)\mathbf{A}$$

$$
\begin{aligned}
&= (I+\mathcal{R})\tilde{\mathbf{\Gamma}}(\mathbf{y},\mathbf{z},t)\mathbf{A}\\
&= [(I+\mathcal{R})K(\mathbf{y}-\mathbf{z}(t),t)]\,e^{-t\Omega}\mathbf{A}.
\end{aligned}
$$

Now, recall that for $f \in \mathcal{S}(\mathbf{R}^3)$

$$\mathcal{R}_i\mathcal{R}_j f = -\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\Delta^{-1} f.$$

([24], pg 243.) It is then natural to consider

$$\Psi(\mathbf{y},\mathbf{z},t) = \frac{1}{4\pi}\frac{1}{(4\pi t)^{3/2}}\int \frac{1}{|\mathbf{y}-\mathbf{x}|}\exp\left(-\frac{|\mathbf{x}-\mathbf{z}(t)|^2}{4t}\right)d\mathbf{x} = -\Delta^{-1}K, \quad (3.8)$$

so that

$$\mathbf{\Gamma}(\mathbf{y},\mathbf{z},t)\mathbf{A} = \left(K(\mathbf{y}-\mathbf{z}(t),t)I + \mathrm{Hess}(\Psi)(\mathbf{y},\mathbf{z},t)\right)e^{-t\Omega}\mathbf{A} \quad (3.9)$$

where Hess(Ψ) stands for the Hessian matrix of Ψ with differentiation taken with respect to the components of $\mathbf{y}$. It should be noted that the formula in (3.9) extends identical formulas for the case of the time dependent Stokes and Oseen problems ([23], [17]).

The first explicit calculation gives Ψ in terms of the Error function

$$\mathrm{Erf}(s) = \frac{2}{\sqrt{\pi}}\int_0^s e^{-u^2}du.$$

Lemma 3.1 *Let $\Psi(\mathbf{y},\mathbf{z},t)$ be defined by (3.8). Then for $t > 0$,*

$$\Psi(\mathbf{y},\mathbf{z},t) = \frac{1}{4\pi|\mathbf{y}-\mathbf{z}(t)|}\mathrm{Erf}\left(\frac{|\mathbf{y}-\mathbf{z}(t)|}{\sqrt{4t}}\right).$$

Proof: Using spherical coordinates with center at $\mathbf{y}$ and azimuthal angle measured from the axis determined by the vector $\mathbf{y}-\mathbf{z}(t)$, one has

$$\Psi(\mathbf{y},\mathbf{z},t) \quad = \quad \frac{1}{32\pi^{5/2}t^{3/2}}\int_0^\infty\int_0^\pi\int_0^{2\pi}\frac{1}{\rho}\exp(-(\rho^2+|\mathbf{y}-\mathbf{z}(t)|^2)/4t)$$

$$
\begin{aligned}
& \exp(-\rho|\mathbf{y}-\mathbf{z}(t)|\cos\phi/2t)\rho^2\sin\phi\, d\theta d\phi d\rho \\
= \; & \frac{2\pi}{32\pi^{5/2}t^{3/2}}\int_0^\infty \exp(-(\rho^2+|\mathbf{y}-\mathbf{z}(t)|^2)/4t) \\
& \frac{2t}{|\mathbf{y}-\mathbf{z}(t)|}[\exp(-\rho|\mathbf{y}-\mathbf{z}(t)|\cos\phi/2t)]|_0^\pi d\rho \\
= \; & \frac{1}{8|\mathbf{y}-\mathbf{z}(t)|\pi^{3/2}t^{1/2}}\left[\int_0^\infty \exp(-(\rho-|\mathbf{y}-\mathbf{z}(t)|)^2)/4t)d\rho\right. \\
& \left. -\int_0^\infty \exp(-(\rho+|\mathbf{y}-\mathbf{z}(t)|)^2)/4t)d\rho\right] \\
= \; & \frac{1}{8|\mathbf{y}-\mathbf{z}(t)|\pi^{3/2}t^{1/2}}\int_{-|\mathbf{y}-\mathbf{z}(t)|}^{|\mathbf{y}-\mathbf{z}(t)|}\exp(-\rho^2/4t)d\rho \\
= \; & \frac{1}{8|\mathbf{y}-\mathbf{z}(t)|\pi^{3/2}t^{1/2}}2\sqrt{4t}\int_0^{|\mathbf{y}-\mathbf{z}(t)|/\sqrt{4t}}\exp(-u^2)du \\
= \; & \frac{1}{8|\mathbf{y}-\mathbf{z}(t)|\pi^{3/2}t^{1/2}}(4\pi t)^{1/2}\mathrm{Erf}(|\mathbf{y}-\mathbf{z}(t)|/\sqrt{4t})
\end{aligned}
$$

and the result follows. ■

Since

$$
\mathrm{Erf}(s) = \frac{2s}{\sqrt{\pi}}e^{-s^2}\ {}_1F_1(1,3/2,s^2), \tag{3.10}
$$

(see [16] page 285), one has

$$
\begin{aligned}
\Psi(\mathbf{y},\mathbf{z},t) &= \frac{1}{4\pi|\mathbf{y}-\mathbf{z}(t)|}\frac{2}{\sqrt{\pi}}\frac{|\mathbf{y}-\mathbf{z}(t)|}{\sqrt{4t}}\exp\left(-|\mathbf{y}-\mathbf{z}(t)|^2/4t\right)\ {}_1F_1(1,\frac{3}{2},\frac{|\mathbf{y}-\mathbf{z}(t)|^2}{4t}) \\
&= 2t\ K(\mathbf{y}-\mathbf{z}(t),t)\ {}_1F_1(1,\frac{3}{2},\frac{|\mathbf{y}-\mathbf{z}(t)|^2}{4t}) \qquad (3.11)
\end{aligned}
$$

Proposition 3.2 *For $t>0$ and with Ψ given by (3.11),*

$$
\begin{aligned}
\frac{\partial}{\partial y_i}\frac{\partial}{\partial y_j}\Psi(\mathbf{y},\mathbf{z},t) &= K(\mathbf{y}-\mathbf{z}(t),t)\left[-\frac{1}{3}\ {}_1F_1\left(1,\frac{5}{2},\frac{|\mathbf{y}-\mathbf{z}(t)|^2}{4t}\right)\delta_{ij}\right. \\
&\left. + \frac{(y_i-z_i(t))(y_j-z_j(t))}{|\mathbf{y}-\mathbf{z}(t)|^2}\left[\ {}_1F_1\left(1,\frac{5}{2},\frac{|\mathbf{y}-\mathbf{z}(t)|^2}{4t}\right)-1\right]\right].
\end{aligned}
$$

Proof: Let $u = |\mathbf{y}-\mathbf{z}(t)|^2/(4t)$ so

$$
\Psi(\mathbf{y},\mathbf{z},t) = \frac{2t}{(4\pi t)^{3/2}}e^{-u}\ {}_1F_1(1,3/2,u).
$$

Consequently,

$$\begin{aligned}\frac{\partial\Psi}{\partial y_i} &= \frac{\partial u}{\partial y_i}\frac{2t}{(4\pi t)^{3/2}}\frac{de^{-u}\ {}_1F_1(1,3/2,u)}{du}\\ &= \frac{\partial u}{\partial y_i}\frac{2t}{(4\pi t)^{3/2}}\left(-\frac{1}{2}\right)\frac{2}{3}e^{-u}\ {}_1F_1(1,5/2,u)\end{aligned}$$

where Lemma 2.1 with $c = 3/2$ was used. Then, since $\partial u/\partial y_i = (y_i - z_i(t))/(2t)$,

$$\frac{\partial\Psi}{\partial y_i} = -\frac{1}{3}(y_i - z_i(t))K(\mathbf{y}-\mathbf{z}(t),t)\ {}_1F_1(1,\frac{5}{2},\frac{|\mathbf{y}-\mathbf{z}(t)|^2}{4t}).$$

For the second derivative, similar arguments can be used. Indeed,

$$\begin{aligned}\frac{\partial^2\Psi}{\partial y_j\partial y_i} &= -\frac{1}{3}\delta_{ij}K(\mathbf{y}-\mathbf{z}(t),t)\ {}_1F_1(1,\frac{5}{2},\frac{|\mathbf{y}-\mathbf{z}(t)|^2}{4t})\\ &\quad -\frac{1}{3}(y_i - z_i(t))\frac{(y_j - z_j(t))}{2t}\frac{1-5/2}{5/2}K(\mathbf{y}-\mathbf{z}(t),t)\ {}_1F_1(1,\frac{7}{2},\frac{|\mathbf{y}-\mathbf{z}(t)|^2}{4t})\\ &= -\frac{1}{3}\delta_{ij}K(\mathbf{y}-\mathbf{z}(t),t)\ {}_1F_1(1,\frac{5}{2},\frac{|\mathbf{y}-\mathbf{z}(t)|^2}{4t})\\ &\quad +\frac{(y_i - z_i(t))(y_j - z_j(t))}{|\mathbf{y}-\mathbf{z}(t)|^2}K(\mathbf{y}-\mathbf{z}(t),t)\\ &\quad \frac{|\mathbf{y}-\mathbf{z}(t)|^2}{4t}\frac{1}{\frac{5}{2}}\ {}_1F_1(1,\frac{7}{2},\frac{|\mathbf{y}-\mathbf{z}(t)|^2}{4t})\end{aligned}$$

The proposition follows using the power series representation of the Kummer functions. Indeed,

$$\begin{aligned}u\frac{2}{5}\ {}_1F_1(1,\frac{7}{2},u) &= u\frac{2}{5}\sum_{n=0}^{\infty}\frac{n!}{(7/2)_n}\frac{u^n}{n!}\\ &= \sum_{n=0}^{\infty}\frac{(n+1)!}{(5/2)_{n+1}}\frac{u^{n+1}}{(n+1)!}\\ &= {}_1F_1(1,\frac{5}{2},u) - 1. \qquad \blacksquare\end{aligned}$$

Proof of Theorem 1.1: The explicit formula for $\mathbf{\Gamma}$ is an immediate consequence of Proposition 3.2 and (3.9). Also, by construction, $\mathbf{\Gamma}(\mathbf{y},\mathbf{z},t)\mathbf{A}$ satisfies for $t > 0$,

$$\frac{\partial \mathbf{\Gamma A}}{\partial t} = \mathcal{L}_{\mathbf{y}}(\mathbf{\Gamma A}).$$

So it only remains to show that

$$\lim_{t \to 0^+} \int_{\mathbf{R}^3} \mathbf{\Gamma}(\mathbf{y}, \mathbf{z}, t)\mathbf{F}(\mathbf{z})d\mathbf{z} + \nabla_{\mathbf{y}} \int_{\mathbf{R}^3} \mathbf{Q}^*(\mathbf{y}, \mathbf{z})\mathbf{F}(\mathbf{z})d\mathbf{z} = \mathbf{F}(\mathbf{z}) \tag{3.12}$$

for all $\mathbf{F} \in (\mathcal{S}(\mathbf{R}^3))^3$.

Using the Helmholtz decomposition one has

$$\mathbf{F} = \mathbf{H} + \nabla V,$$

where, for example, $\mathbf{H}$ and V are in $L^2(\mathbf{R}^3)$ and $\nabla \cdot \mathbf{H} = 0$. Then, since $\mathbf{z}(0) = \mathbf{z}$,

$$\begin{aligned}
\int_{\mathbf{R}^3} \mathbf{Q}^*(\mathbf{y}, \mathbf{z})\mathbf{F}(\mathbf{z}) &= \int_{\mathbf{R}^3} \frac{-1}{4\pi} \nabla_{\mathbf{y}} \frac{1}{|\mathbf{y} - \mathbf{z}|} \nabla V(\mathbf{z})d\mathbf{z} \\
&= \int_{\mathbf{R}^3} \frac{1}{4\pi} \nabla_{\mathbf{z}} \frac{1}{|\mathbf{y} - \mathbf{z}|} \nabla V(\mathbf{z})d\mathbf{z} \\
&= -\int_{\mathbf{R}^3} \frac{1}{4\pi} \frac{1}{|\mathbf{y} - \mathbf{z}|} \Delta V(\mathbf{z})d\mathbf{z} \\
&= V(\mathbf{y})
\end{aligned}$$

so

$$\nabla_{\mathbf{y}} \int_{\mathbf{R}^3} \mathbf{Q}^*(\mathbf{y}, \mathbf{z})\mathbf{F}(\mathbf{z}) = \nabla_{\mathbf{y}} V(\mathbf{y}). \tag{3.13}$$

Using Lemma 3.1, and $\mathbf{z}(t) = e^{-t\Omega}\mathbf{z} - t\mathbf{U}$ it is easy to see that

$$\mathrm{Hess}_{\mathbf{y}}(\Psi) = e^{-t\Omega} \mathrm{Hess}_{\mathbf{z}}(\Psi) e^{t\Omega}.$$

Then, using (3.9),

$$\begin{aligned}
\int_{\mathbf{R}^3} \mathbf{\Gamma}(\mathbf{y}, \mathbf{z}, t)\mathbf{F}(\mathbf{z})d\mathbf{z} &= e^{-t\Omega} \int_{\mathbf{R}^3} K(\mathbf{y} - \mathbf{z}(t), t)\mathbf{F}(\mathbf{z})d\mathbf{z} \\
&\quad + e^{-t\Omega} \int_{\mathbf{R}^3} \mathrm{Hess}_{\mathbf{z}}(\Psi)(\mathbf{y}, \mathbf{z}, t)\mathbf{F}(\mathbf{z}))d\mathbf{z}
\end{aligned}$$

Standard properties of the three dimensional heat kernel give

$$\lim_{t\to 0^+} e^{-t\Omega}\int_{\mathbf{R}^3} K(\mathbf{y}-\mathbf{z}(t),t)\mathbf{F}(\mathbf{z})d\mathbf{z} = \mathbf{F}(\mathbf{y}). \tag{3.14}$$

Now, recall that the Laplace operator and its inverse are invariant under orthogonal transformation, so working componentwise with $i=1,2,3$ one has

$$\begin{aligned}\int_{\mathbf{R}^3} (\mathrm{Hess}_{\mathbf{z}}(\Psi)(\mathbf{y},\mathbf{z},t)\mathbf{F}(\mathbf{z}))_i d\mathbf{z} &= \int_{\mathbf{R}^3} \nabla_{\mathbf{z}}\left(\frac{\partial\Psi}{\partial z_i}\right)\cdot\mathbf{F}(\mathbf{z})d\mathbf{z}\\ &= \int_{\mathbf{R}^3} \Psi\frac{\partial\nabla_{\mathbf{z}}\cdot\mathbf{F}}{\partial z_i}d\mathbf{z}\\ &= \int_{\mathbf{R}^3} \Psi\frac{\partial\Delta V}{\partial z_i}d\mathbf{z}\\ &= \int_{\mathbf{R}^3} -\Delta^{-1}K(\mathbf{y}-\mathbf{z}(t),t)\Delta\frac{\partial V}{\partial z_i}d\mathbf{z}\\ &= -\int_{\mathbf{R}^3} K(\mathbf{y}-\mathbf{z}(t),t)\frac{\partial V}{\partial z_i}d\mathbf{z}, \qquad (3.15)\end{aligned}$$

where we have used the Helmholtz decomposition of $\mathbf{F}$. In particular, for $\mathbf{F}\in(\mathcal{S}(\mathbf{R}^3))^3$,

$$\int_{\mathbf{R}^3}\mathbf{\Gamma}(\mathbf{y},\mathbf{z},t)\mathbf{F}(\mathbf{z})d\mathbf{z} = \int_{\mathbf{R}^3} K(\mathbf{y}-\mathbf{z}(t),t)\mathbf{H}(\mathbf{z})dz. \tag{3.16}$$

Thus

$$\begin{aligned}\lim_{t\to 0^+}\int_{\mathbf{R}^3}(\mathrm{Hess}_{\mathbf{y}}(\Psi)(\mathbf{y},\mathbf{z},t)\mathbf{F}(\mathbf{z}))_i dz &= \lim_{t\to 0^+} -e^{-t\Omega}\int_{\mathbf{R}^3} K(\mathbf{y}-\mathbf{z}(t),t)\frac{\partial V}{\partial z_i}d\mathbf{z}\\ &= -\frac{\partial V}{\partial y_i}(\mathbf{y}) \qquad (3.17)\end{aligned}$$

Then (3.12) follows from (3.13), (3.14) and (3.17). ■

4 Basic Properties of the Fundamental Solution

In this section, the asymptotic behavior of the fundamental solution $\mathbf{\Gamma}(\mathbf{y},\mathbf{z},t)$ and estimates of its spatial and time derivatives are obtained. While more general estimates are possible, the results presented here are correspond to the spatial derivatives of up to second order and the first time derivative. The proofs of Theorem 1.2 and Theorem 1.3 are also given in this section.

A few observations regarding the structure of the fundamental solution are in order. Let $u = |\mathbf{y} - \mathbf{z}(t)|^2/4t$, $\theta = (\mathbf{y} - \mathbf{z}(t))/|\mathbf{y} - \mathbf{z}(t)|$, and $\Lambda(\theta) = (\theta \otimes \theta)$. Let

$$\mathbf{M}(\mathbf{y}, \mathbf{z}, t) = \frac{1}{3}\frac{1}{(4\pi t)^{3/2}} e^{-u} \;{}_1F_1(1, 5/2, u)[I - 3\Lambda(\theta)], \tag{4.1}$$

so that

$$\mathbf{\Gamma} = [K(\mathbf{y} - \mathbf{z}(t), t)[I - \Lambda(\theta)] - M(\mathbf{y}, \mathbf{z}, t)]\, e^{-t\Omega}.$$

Estimates on $\mathbf{\Gamma}$ can be obtained combining known estimates for the heat kernel and new estimates for $\mathbf{M}$. Note also that since as a function of $\mathbf{y} - \mathbf{z}(t)$, Λ is a matrix of homogeneous functions of degree zero, its derivatives satisfy

$$\left| \frac{\partial^{|\alpha|}\Lambda}{\partial \mathbf{y}^{\alpha}} \right| \leq \frac{C}{|\mathbf{y} - \mathbf{z}(t)|^{|\alpha|}}. \tag{4.2}$$

Here, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ denotes a multiindex of nonnegative integers, $|\alpha| = \sum_{i=1}^{3} \alpha_i$, and the usual multiindex notation for derivatives has been used.

The first result establishes the *asymptotic behavior* of the fundamental solution.

Proposition 4.1 *Let* $\mathbf{y}^0$, $\mathbf{z}, \eta \in \mathbf{R}^3$, $|\eta| = 1$, *be fixed and let* $\mathbf{y} = \mathbf{y}^0 + \rho\eta$, $\rho > 0$. *Then for fixed* $t > 0$

$$\begin{aligned} \mathbf{\Gamma}(\mathbf{y}, \mathbf{z}, t) &\sim -\frac{1}{4\pi}\frac{1}{|\mathbf{y} - \mathbf{z}(t)|^3}[I - 3\eta \otimes \eta]e^{-t\Omega} \quad \text{as} \quad \rho \to \infty, \\ \mathbf{\Gamma}(\mathbf{y}, \mathbf{z}, t) &\sim \frac{1}{(4\pi t)^{3/2}}\frac{2}{3}e^{-t\Omega} \quad \text{as} \quad \rho \to 0 \quad \text{and} \quad \mathbf{z} \to \mathbf{y}^0, \end{aligned}$$

whereas for ρ fixed,

$$\mathbf{\Gamma}(\mathbf{y}, \mathbf{z}, t) \sim -\frac{1}{4\pi} \frac{1}{|\mathbf{y} - \mathbf{z}|^3} \left[I - 3 \frac{(\mathbf{y} - \mathbf{z}) \otimes (\mathbf{y} - \mathbf{z})}{|\mathbf{y} - \mathbf{z}|^2} \right] \quad \text{as} \quad t \to 0.$$

Proof: Consider first the cases $\rho \to \infty$ and $t \to 0$. Using the known asymptotic behavior of the Kummer functions noted in (2.4), one has that in both cases,

$$\begin{aligned} K(\mathbf{y} - \mathbf{z}(t), t) \ {}_1F_1(1, 5/2, |\mathbf{y} - \mathbf{z}(t)|^2/4t) &= \frac{1}{(4\pi t)^{3/2}} e^{-u} \ {}_1F_1(1, 5/2, u) \\ &\sim \frac{1}{(4\pi t)^{3/2}} \Gamma(5/2) \left(\frac{4t}{|\mathbf{y} - \mathbf{z}(t)|^2} \right)^{3/2} \\ &= \frac{3}{4\pi} \frac{1}{|\mathbf{y} - \mathbf{z}(t)|^3}. \end{aligned} \tag{4.3}$$

Note that

$$\Lambda(\theta) \sim \eta \otimes \eta \quad \text{as} \quad \rho \to \infty, \ t \ \text{ fixed} \tag{4.4}$$

$$\Lambda(\theta) \sim \frac{(\mathbf{y} - \mathbf{z}) \otimes (\mathbf{y} - \mathbf{z})}{|\mathbf{y} - \mathbf{z}|^2} \quad \text{as} \quad t \to 0, \ \rho \ \text{ fixed.} \tag{4.5}$$

Since for $\mathbf{y} \neq \mathbf{z}$, $\lim_{t \to 0} K(\mathbf{y} - \mathbf{z}(t), t) = 0$, and for t fixed, $\lim_{\mathbf{y} \to \infty} K(\mathbf{y} - \mathbf{z}(t), t) = 0$, the proposition follows in these cases from (4.3), (4.4) and (4.5.)

Finally, the case $\rho \to 0, \mathbf{z} \to \mathbf{y}^0$ is immediate, since $\lim_{u \to 0} \ {}_1F_1(1, 5/2, u) = 1$. ■

The asymptotic behavior of the fundamental solution for large t is also noted as it may be of use in the study of steady state problems.

Proposition 4.2 *Let $\mathbf{y}, \mathbf{z} \in \mathbf{R}^3$ be fixed. Then, as $t \to \infty$,*

$$\begin{aligned} e^{t\Omega} \mathbf{\Gamma}(\mathbf{y}, \mathbf{z}, t) &\sim \frac{1}{(4\pi t)^{3/2}} \frac{2}{3} I \quad \text{if} \quad \mathbf{U} = 0, \\ e^{t\Omega} \mathbf{\Gamma}(\mathbf{y}, \mathbf{z}, t) &\sim -\frac{1}{4\pi} \frac{1}{(t|\mathbf{U}|)^3} \left[I - 3 \frac{\mathbf{U} \otimes \mathbf{U}}{|\mathbf{U}|^2} \right] \quad \text{if} \quad \mathbf{U} \neq 0. \end{aligned}$$

Remark 2 *As a corollary of Proposition 4.2 one obtains the asymptotic behavior of the fundamental solution of the time dependent Stokes and Oseen problems since they correspond to* $\omega = 0$. *Denote by* $\mathbf{\Gamma}_{\mathcal{S}}$ *and* $\mathbf{\Gamma}_{\mathcal{O}}$ *the respective fundamental solution. Then as* $t \to \infty$,

$$\mathbf{\Gamma}_{\mathcal{S}}(\mathbf{y},\mathbf{z},t) \sim \frac{1}{(4\pi t)^{3/2}}\frac{2}{3}I, \qquad \mathbf{\Gamma}_{\mathcal{O}}(\mathbf{y},\mathbf{z},t) \sim \frac{1}{4\pi}\frac{1}{(t|\mathbf{U}|)^3}\left[I - 3\frac{\mathbf{U}\otimes\mathbf{U}}{|\mathbf{U}|^2}\right].$$

Proof: Consider first the case $\mathbf{U} = 0$. Then, as $t \to \infty$, $u = |\mathbf{y} - e^{-t\Omega}\mathbf{z}|^2/4t \to 0$ and thus

$$e^{-u}\left[1 - {}_1F_1(1,5/2,u)\right] \to 0$$

as well. Since

$$\begin{aligned} e^{t\Omega}\mathbf{\Gamma} &= \frac{1}{(4\pi t)^{3/2}}e^{-u}\left(1 - \frac{1}{3}\,{}_1F_1(1,5/2,u)\right)I \\ &\quad -\frac{1}{(4\pi t)^{3/2}}e^{-u}\left(1 - {}_1F_1(1,5/2,u)\right)\Lambda(\theta), \end{aligned}$$

the asymptotic behavior for large t is given by the first term and the proposition follows in this case.

Let $\mathbf{U} \neq 0$, and recall that $\mathbf{z}(t) = e^{-t\Omega}\mathbf{z} - t\mathbf{U}$. Then it follows that for large t,

$$u = |\mathbf{y} - \mathbf{z}(t)|^2/4t \sim t|\mathbf{U}|^2/4, \quad \text{and} \quad \theta = \frac{\mathbf{y} - \mathbf{z}(t)}{|\mathbf{y} - \mathbf{z}(t)|} \sim \frac{\mathbf{U}}{|\mathbf{U}|}.$$

Now, using (2.4), one has

$$\begin{aligned} &K(\mathbf{y} - \mathbf{z}(t),t)\ {}_1F_1(1,5/2,|\mathbf{y} - \mathbf{z}(t)|^2/4t)\frac{1}{3}[I - 3\Lambda(\theta)] \\ &\quad\sim \frac{1}{3}\frac{\Gamma(5/2)}{(4\pi t)^{3/2}}\left(\frac{4}{t|\mathbf{U}|^2}\right)^{3/2}\left[I - 3\frac{\mathbf{U}\otimes\mathbf{U}}{|\mathbf{U}|^2}\right] \\ &\quad= \frac{1}{4\pi}\frac{1}{(t|\mathbf{U}|)^3}\left[I - 3\frac{\mathbf{U}\otimes\mathbf{U}}{|\mathbf{U}|^2}\right]. \end{aligned} \tag{4.6}$$

On the other hand,

$$K(\mathbf{y}-\mathbf{z}(t),t) \sim \frac{e^{-t|\mathbf{U}|^2/4}}{(4\pi t)^{3/2}} e^{-((\mathbf{y}-e^{-t\Omega}\mathbf{z})\cdot\mathbf{U})/2},$$

which is of lower order than (4.6) and the proposition follows. ■

The main estimates for the derivatives of $\mathbf{\Gamma}$ are contained in the next proposition. To obtain them, it is convenient to introduce for fixed T, the sets

$$\begin{aligned} \mathcal{M}_0 &= \{(\mathbf{x},t) : 0 \le t \le T, |\mathbf{x}|^2/t > 1\}, \\ \mathcal{M}_1 &= \{(\mathbf{x},t) : 0 \le t \le T, |\mathbf{x}| \ge 1, |\mathbf{x}|^2/t \le 1\}, \quad \text{and} \\ \mathcal{M}_2 &= \{(\mathbf{x},t) : 0 \le t \le T, |\mathbf{x}| < 1, |\mathbf{x}|^2/t \le 1\}, \end{aligned}$$

and to denote by $\chi_j(\mathbf{x},t)$, $j = 0,1,2$, the characteristic function of the set $\mathcal{M}_j$. Note that these sets give a partition of the space $\mathbf{R}^3 \times [0,T]$.

Proposition 4.3 *Let* $\mathbf{y} \in \mathbf{R}^3$ *be fixed but arbitrary. Given* $\alpha = (\alpha_1,\alpha_2,\alpha_3)$, $|\alpha| \le 2$, $T > 0$, *and* $\lambda > 1$, *there exist nonnegative constants* $C_{\lambda l}$, $C_{kl}, k, l = 0,1,2,$ *such that with* $a_0 = 0, a_1 = a_2 = 1,$ *and*

$$h_l(\mathbf{x},t) = \frac{C_{0l}}{|\mathbf{x}|^{3+l}}\chi_0(\mathbf{x},t) + C_{1l}\chi_1(\mathbf{x},t) + \frac{C_{2l}}{t^{3/2}|\mathbf{x}|^l}\chi_2(\mathbf{x},t) + \frac{C_{\lambda l}}{t^{l/2}}K(\mathbf{x},(\lambda)^{a_l}t),$$

$$\left|\frac{\partial^{|\alpha|}\mathbf{\Gamma}_{ij}(\mathbf{y},\mathbf{z},t)}{\partial \mathbf{y}^\alpha}\right| \le h_{|\alpha|}(\mathbf{y}-\mathbf{z}(t),t).$$

for all $0 \le t \le T$, *and* $1 \le i,j \le 3$,

Proof: The case $|\alpha| = 0$ can be obtained using the proof of Proposition 4.1. Note also that $\mathbf{\Gamma}$ and its derivatives are continuous and thus bounded on the compact set $\mathcal{M}_1$.

It is easy to obtain estimates for the terms involving only the heat kernel using well known results. For example, from Lemma 1-4, Chapter 9, in [13], one has with $\lambda > 1$,

$$\begin{aligned}\left|\frac{\partial K(\mathbf{y}-\mathbf{z}(t),t)}{\partial y_l}\right| &\leq \frac{C_\lambda}{\sqrt{t}}K(\mathbf{y}-\mathbf{z}(t),\lambda t)\\ \left|\frac{\partial^2 K(\mathbf{y}-\mathbf{z}(t),t)}{\partial y_l \partial y_m}\right| &\leq \frac{C_\lambda}{t}K(\mathbf{y}-\mathbf{z}(t),\lambda t).\end{aligned}$$

Therefore,

$$\left|\frac{\partial^{|\alpha|} K(\mathbf{y}-\mathbf{z}(t),t)}{\partial \mathbf{y}^\alpha}\right| \leq \frac{C_{\lambda|\alpha|}}{t^{|\alpha|/2}}K(\mathbf{y}-\mathbf{z}(t),\lambda t). \tag{4.7}$$

Recall that $\theta = (\mathbf{y}-\mathbf{z}(t))/|\mathbf{y}-\mathbf{z}(t)|$, $u = |\mathbf{y}-\mathbf{z}(t)|^2/4t$ and $\mathbf{M}$ is defined in (4.1). From Lemma 2.1 one has

$$\begin{aligned}\frac{\partial \mathbf{M}}{\partial y_l} &= \frac{1}{(4\pi t)^{3/2}}\frac{\partial u}{\partial y_l}\left[\frac{1}{3}\frac{1-5/2}{5/2}e^{-u}\ {}_1F_1(1,7/2,u)[I-3\Lambda]\right]\\ &\quad -\frac{1}{(4\pi t)^{3/2}}e^{-u}\ {}_1F_1(1,5/2,u)\frac{\partial \Lambda}{\partial y_l}\end{aligned}$$

so that

$$\begin{aligned}e^{t\Omega}\frac{\partial \mathbf{\Gamma}}{\partial y_l} &= \frac{\partial K(\mathbf{y}-\mathbf{z}(t),t)}{\partial y_l}[I-\Lambda] + \frac{1}{5}\frac{\partial u}{\partial y_l}\frac{1}{(4\pi t)^{3/2}}e^{-u}\ {}_1F_1(1,7/2,u)[I-3\Lambda]\\ &\quad +\frac{1}{(4\pi t)^{3/2}}e^{-u}(\ {}_1F_1(1,5/2,u)-1)\frac{\partial \Lambda}{\partial y_l}\\ &= \frac{\partial K(\mathbf{y}-\mathbf{z}(t),t)}{\partial y_l}[I-\Lambda] + \frac{1}{5}\frac{\partial u}{\partial y_l}\frac{1}{(4\pi t)^{3/2}}e^{-u}\ {}_1F_1(1,7/2,u)[I-3\Lambda] \qquad (4.8)\\ &\quad +\frac{1}{(4\pi t)^{3/2}}\frac{2}{5}ue^{-u}\ {}_1F_1(1,7/2,u)\frac{\partial \Lambda}{\partial y_l} \equiv \mathbf{\Gamma}_0(\mathbf{y},\mathbf{z},t)+\mathbf{\Gamma}_1(\mathbf{y},\mathbf{z},t)+\mathbf{\Gamma}_2(\mathbf{y},\mathbf{z},t).\end{aligned}$$

In the last step of this calculation, the identity

$${}_1F_1(1,c,u) - 1 = \frac{1}{c}u\ {}_1F_1(1,c+1,u) \tag{4.9}$$

valid for $c > 0$ has been used. Also, the terms involving $\partial\Lambda/\partial y_l$ have been combined resulting in $\mathbf{\Gamma}_2$ to take advantage of the cancellation of terms occurring for $\mathbf{z}(t)$ close to $\mathbf{y}$.

Note that

$$\left|\frac{\partial u}{\partial y_l}\right| = \left|\frac{y_l - z_l(t)}{2t}\right| \leq \frac{|\mathbf{y} - \mathbf{z}(t)|}{2t}.$$

From (2.4), it follows that on $\mathcal{M}_0$,

$$\begin{aligned} |\mathbf{\Gamma}_1(\mathbf{y}, \mathbf{z}, t)| &\leq C\left|\frac{\partial u}{\partial y_l}\frac{1}{t^{3/2}}e^{-u}\ {}_1F_1(1, 7/2, u)\right| \\ &\leq C\frac{|\mathbf{y} - \mathbf{z}(t)|}{t}\frac{1}{t^{3/2}}\left(\frac{t}{|\mathbf{y} - \mathbf{z}(t)|^2}\right)^{5/2} \\ &= C\frac{1}{|\mathbf{y} - \mathbf{z}(t)|^4}. \end{aligned} \tag{4.10}$$

Similarly, recall that from (4.2),

$$\left|\frac{\partial \Lambda}{\partial y_l}\right| \leq \frac{C}{|\mathbf{y} - \mathbf{z}(t)|}. \tag{4.11}$$

Then, from (2.4) one has

$$\begin{aligned} |\mathbf{\Gamma}_2(\mathbf{y}, \mathbf{z}, t)| &\leq C\left|\frac{\partial \Lambda}{\partial y_l}u\frac{1}{t^{3/2}}e^{-u}\ {}_1F_1(1, 7/2, u)\right| \\ &\leq C\frac{1}{|\mathbf{y} - \mathbf{z}(t)|}\frac{|\mathbf{y} - \mathbf{z}(t)|^2}{t}\frac{1}{t^{3/2}}\left(\frac{t}{|\mathbf{y} - \mathbf{z}(t)|^2}\right)^{5/2} \\ &= C\frac{1}{|\mathbf{y} - \mathbf{z}(t)|^4}. \end{aligned} \tag{4.12}$$

Therefore, the estimates on $\mathcal{M}_0$ for the first derivatives follow from (4.7), (4.10) and (4.12).

On $\mathcal{M}_2$, $e^{-u}\ {}_1F_1(1, c, u)$ is bounded and

$$\left|\frac{\partial u}{\partial y_l}\right| \leq \frac{|\mathbf{y} - \mathbf{z}(t)|}{2t} = \frac{1}{2|\mathbf{y} - \mathbf{z}(t)|}\frac{|\mathbf{y} - \mathbf{z}(t)|^2}{t} \leq \frac{1}{2|\mathbf{y} - \mathbf{z}(t)|}.$$

Then,

$$|\mathbf{\Gamma}_1(\mathbf{y}, \mathbf{z}, t)| \leq C\frac{1}{|\mathbf{y} - \mathbf{z}(t)|}\frac{1}{t^{3/2}}.$$

A similar estimate holds for $\mathbf{\Gamma}_2$ by (4.11) so the proposition follows for $|\alpha| = 1$.

In preparation for the estimates of the second derivatives of the fundamental solution, note that

$$\begin{aligned}\left[-\frac{\partial K}{\partial y_l}-\frac{3}{5}\frac{\partial u}{\partial y_l}\frac{1}{(4\pi t)^{3/2}}e^{-u}\ {}_1F_1(1,7/2,u)\right]\Lambda &= \frac{\partial u}{\partial y_l}\left[\frac{1}{(4\pi t)^{3/2}}e^{-u}\right.\\ -\frac{1}{(4\pi t)^{3/2}}e^{-u}\ {}_1F_1(1,7/2,u) &+ \left.\frac{2}{5}\frac{1}{(4\pi t)^{3/2}}e^{-u}\ {}_1F_1(1,7/2,u)\right]\Lambda\\ =\frac{\partial u}{\partial y_l}\left[\frac{2}{5}\frac{1}{(4\pi t)^{3/2}}e^{-u}\ {}_1F_1(1,7/2,u)\right. &- \left.\frac{2}{7}\frac{u}{(4\pi t)^{3/2}}e^{-u}\ {}_1F_1(1,9/2,u)\right]\Lambda,\end{aligned}$$

where in the last step we have used (4.9) with $c = 7/2$.

Combining the terms involving Λ in (4.8), one has

$$\begin{aligned}e^{t\Omega}\frac{\partial\mathbf{\Gamma}}{\partial y_l} &= \frac{\partial K}{\partial y_l}I+\frac{1}{5}\frac{\partial u}{\partial y_l}\frac{1}{(4\pi t)^{3/2}}e^{-u}\ {}_1F_1(1,7/2,u)I\\ &+\frac{\partial u}{\partial y_l}\left[\frac{2}{5}\frac{1}{(4\pi t)^{3/2}}e^{-u}\ {}_1F_1(1,7/2,u)-\frac{2}{7}\frac{u}{(4\pi t)^{3/2}}e^{-u}\ {}_1F_1(1,9/2,u)\right]\Lambda\\ &+\frac{2}{5}\frac{u}{(4\pi t)^{3/2}}e^{-u}\ {}_1F_1(1,7/2,u)\frac{\partial\Lambda}{\partial y_l}\end{aligned}$$

The second derivatives can be written using Lemma 2.1 and with appropriate constants $a_p^{(q)}, b_p, c_p, d_0$ for $p, q = 0, 1$ as

$$\begin{aligned}e^{t\Omega}\frac{\partial^2\mathbf{\Gamma}}{\partial y_l\partial y_m} &= \frac{\partial^2 K}{\partial y_l\partial y_m}I\\ &+\frac{1}{t^{3/2}}\frac{\partial^2 u}{\partial y_l\partial y_m}\left[\sum_{p,q=0}^{1}a_p^{(q)}u^p e^{-u}\ {}_1F_1(1,7/2+p,u)[qI+(1-q)\Lambda]\right]\\ &+\frac{1}{t^{3/2}}\frac{\partial u}{\partial y_l}\frac{\partial u}{\partial y_m}\left[\sum_{p=0}^{1}b_p u^p e^{-u}\ {}_1F_1(1,9/2+p,u)\right]\Lambda\\ &+\frac{1}{t^{3/2}}\frac{\partial u}{\partial y_l}\left[\sum_{p=0}^{1}c_p u^p e^{-u}\ {}_1F_1(1,7/2+p,u)\right]\frac{\partial\Lambda}{\partial y_m}\\ &+\frac{1}{t^{3/2}}\frac{\partial u}{\partial y_m}\left[\sum_{p=0}^{1}c_p u^p e^{-u}\ {}_1F_1(1,7/2+p,u)\right]\frac{\partial\Lambda}{\partial y_l}\\ &+\frac{1}{t^{3/2}}d_0 u e^{-u}\ {}_1F_1(1,7/2,u)\frac{\partial^2\Lambda}{\partial y_l\partial y_m}.\end{aligned}$$

Similar arguments as those applied to estimate the first derivatives yield the desired estimate for the second derivative on $\mathcal{M}_0$. For example, on this set

$$\begin{aligned}\left|\frac{1}{t^{3/2}}\frac{\partial u}{\partial y_l}\frac{\partial u}{\partial y_m}u^p e^{-u}\ {}_1F_1(1,9/2+p,u)\right| &\leq C\frac{1}{t^{3/2}}\frac{|\mathbf{y}-\mathbf{z}(t)|^2}{t^2}\left(\frac{|\mathbf{y}-\mathbf{z}(t)|^2}{t}\right)^p \\ \times\left(\frac{t}{|\mathbf{y}-\mathbf{z}(t)|^2}\right)^{p+9/2-1} &= C\frac{1}{|\mathbf{y}-\mathbf{z}(t)|^5}.\end{aligned}$$

Recall that the estimate on $\mathcal{M}_1$ follows by continuity of the derivatives of $\mathbf{\Gamma}$ since this set is compact. So only the behavior of the second derivatives of $\mathbf{\Gamma}$ on the set $\mathcal{M}_2$ remain to be analyzed. Note that on this set, $u^p e^{-u}\ {}_1F_1(1,c+p,u)$ is bounded for $p=0,1$ and $c=7/2,9/2$. Also, on $\mathcal{M}_1$,

$$\left|\frac{\partial^2 u}{\partial y_l \partial y_m}\right| \leq \frac{C}{t} \leq \frac{C}{|\mathbf{y}-\mathbf{z}(t)|^2},$$

and

$$\left|\frac{\partial u}{\partial y_l}\frac{\partial u}{\partial y_m}\right| \leq \frac{C}{|\mathbf{y}-\mathbf{z}(t)|^2}, \quad \left|\frac{\partial u}{\partial y_l}\frac{\partial \Lambda}{\partial y_m}\right| \leq \frac{C}{|\mathbf{y}-\mathbf{z}(t)|^2}.$$

Finally, using (4.2)

$$\left|\frac{\partial^2 \Lambda}{\partial y_l \partial y_m}\right| \leq \frac{C}{t} \leq \frac{C}{|\mathbf{y}-\mathbf{z}(t)|^2},$$

and the proposition follows easily. ■

An immediate consequence of this proposition is the corresponding estimate for the time derivatives.

Corollary 4.1 *Let $T>0$. There exist constants $C_j, j=0,1,2$ such that*

$$\left|\frac{\partial \mathbf{\Gamma}_{ij}}{\partial t}\right| \leq C_0|\omega|h_0 + C_1|\mathbf{U}|h_1 + C_2 h_2.$$

Similarly, integrability properties of the fundamental solution follow immediately from the estimates in Proposition 4.3.

Corollary 4.2 *For fixed $t > 0$, $1 \leq i, j \leq 3$, and either $\mathbf{z}$ or $\mathbf{y}$ fixed,*

$$\mathbf{\Gamma}_{ij}(\mathbf{y}, \mathbf{z}, t) \in L^q(\mathbf{R}^3)$$

for all $1 < q < \infty$.

Proof of Theorems 1.2 and 1.3: In light of the estimates obtained on the fundamental solution in Proposition 4.3, the proof of these theorems follow standard arguments. Recall that $\mathbf{v}$ is given by

$$\mathbf{v}(\mathbf{y}, t) = \int_{\mathbf{R}^3} \mathbf{\Gamma}(\mathbf{y}, \mathbf{z}, t)\mathbf{u}_0(\mathbf{z})d\mathbf{z} + \lim_{\epsilon \to 0} \int_{\epsilon}^{t} \int_{\mathbf{R}^3} \mathbf{\Gamma}(\mathbf{y}, \mathbf{z}, s)\mathbf{F}(\mathbf{z}, t-s)d\mathbf{z}ds.$$

Let $\mathbf{F}_n \in (\mathcal{S}(\mathbf{R}^3))^3$ be a sequence such that

$$\lim_{n \to \infty} |\mathbf{F}(\mathbf{z}, s) - \mathbf{F}_n(\mathbf{z}, s)|_{(L^p(\mathbf{R}^3))^3} = 0$$

for all $0 \leq s \leq t$. Similarly, approximate the initial data $\mathbf{u}_0$ with a sequence $\mathbf{u}_n^{(0)}$. Let

$$\mathbf{v}_{\epsilon,n}(\mathbf{y}, t) = \int_{\mathbf{R}^3} \mathbf{\Gamma}(\mathbf{y}, \mathbf{z}, t)\mathbf{u}_n^{(0)}(\mathbf{z})d\mathbf{z} + \int_{\epsilon}^{t} \int_{\mathbf{R}^3} \mathbf{\Gamma}(\mathbf{y}, \mathbf{z}, s)\mathbf{F}_n(\mathbf{z}, t-s)d\mathbf{z}ds.$$

It follows from Corollary 4.2 that pointwise,

$$\mathbf{v}(\mathbf{y}, t) = \lim_{\epsilon \to 0} \lim_{n \to \infty} \mathbf{v}_{\epsilon,n}.$$

On the other hand, using (3.16) from the proof of Theorem 1.1, one has

$$\mathbf{v}_{n,\epsilon}(\mathbf{y}, t) = \int_{\mathbf{R}^3} K(\mathbf{y}-\mathbf{z}(t), t)\mathcal{P}\mathbf{u}_n^{(0)}(\mathbf{z})d\mathbf{z} + \int_{\epsilon}^{t} \int_{\mathbf{R}^3} K(\mathbf{y}-\mathbf{z}(s), s)\mathcal{P}\mathbf{F}_n(\mathbf{z}, t-s)d\mathbf{z}ds.$$

Since the Helmholtz decomposition is continuous on $L^p(\mathbf{R}^3)$ for $p > 1$, it follows that

$$\mathbf{v}(\mathbf{y}, t) = \int_{\mathbf{R}^3} K(\mathbf{y} - \mathbf{z}(t), t)\mathcal{P}\mathbf{u}_0(\mathbf{z})d\mathbf{z} + \int_{0}^{t} \int_{\mathbf{R}^3} K(\mathbf{y} - \mathbf{z}(s), s)\mathcal{P}\mathbf{F}(\mathbf{z}, t-s)d\mathbf{z}ds.$$

Thus $\mathbf{v}$ satisfies the parabolic equation

$$\frac{\partial \mathbf{v}}{\partial t} = (\mathbf{U} + \omega \wedge \mathbf{y}) \cdot \nabla \mathbf{v} - \omega \wedge \mathbf{v} + \Delta \mathbf{v} + \mathcal{P}(\mathbf{F}) \tag{4.13}$$

with initial data $\mathbf{v}(\mathbf{y}, 0) = \mathcal{P}\mathbf{u}_0(\mathbf{y})$. Using the method of proof from [8], it follows that (4.13) has a unique solution and Theorem 1.2 follows.

Similarly, the semigroup property of $\mathbf{\Gamma}$ follows once one note that for arbitrary $\mathbf{u}_0 \in (\mathcal{S}(\mathbf{R}^3))^3, \nabla \cdot \mathbf{u}_0 = 0$,

$$\mathbf{v}_1(\mathbf{y}, t+s) = \int_{\mathbf{R}^3} \mathbf{\Gamma}(\mathbf{y}, \mathbf{z}, t) \int_{\mathbf{R}^3} \mathbf{\Gamma}(\mathbf{z}, \mathbf{z}', s) \mathbf{u}_0(\mathbf{z}') d\mathbf{z}' d\mathbf{z}$$

and

$$\mathbf{v}_2(\mathbf{y}, t+s) = \int_{\mathbf{R}^3} \mathbf{\Gamma}(\mathbf{y}, \mathbf{z}, t) \mathbf{u}_0(\mathbf{z}) d\mathbf{z}$$

satisfy (4.13) with the same initial data. Then by uniqueness of solutions, the semigroup property follows immediately.

It remains to show that $\mathbf{w}_{\mathbf{A}}(\mathbf{z}, s) = \mathbf{\Gamma}'(\mathbf{y}, \mathbf{z}, t-s)\mathbf{A}$ satisfies the adjoint equation. For this, it is convenient to keep track of the dependence of $\mathbf{\Gamma}$ on $\mathbf{U}$ and ω and write instead $\mathbf{\Gamma}(\mathbf{y}, \mathbf{z}, t-s; \mathbf{U}, \omega)$. Then, the fundamental solution of the adjoint equation is given by $\mathbf{\Gamma}(\mathbf{z}, \mathbf{y}, t-s; -\mathbf{U}, -\omega)$. It follows by inspection that

$$\mathbf{\Gamma}(\mathbf{z}, \mathbf{y}, t-s; -\mathbf{U}, -\omega) = \mathbf{\Gamma}'(\mathbf{y}, \mathbf{z}, t-s; \mathbf{U}, \omega)$$

as claimed. ∎

5 Conclusions and Remarks

The main result of this paper is the determination of the fundamental solution of the linearized Navier-Stokes equations that are applicable to the modeling of rigid bodies that are translating and rotating at constant translational and angular velocities. The knowledge of the fundamental solution could aid in the development of numerical methods based on boundary element methods and in the calculation of the hydrodynamic forces acting on rigid bodies.

The formula obtained gives insight into the large time and space asymptotic and could lead into understanding characteristics of the steady state solutions that are applicable to sedimentation problems. Specifically, the fundamental solution of the steady problem is given by

$$\mathcal{K}(\mathbf{y},\mathbf{z}) = \int_0^\infty \mathbf{\Gamma}(\mathbf{y},\mathbf{z},t)dt.$$

In the case that the angular velocity vanishes, the usual Stokes and Oseen fundamental solution tensors can be obtained in this way. On the other hand, when the angular velocity does not vanish, this formula can be used to estimate the steady state solution and its asymptotic properties.

Finally, it should be noted that the explicit formula for the fundamental solution constitutes a first step towards establishing a probabilistic description of the solution of incompressible fluid flows. A probabilistic description is available using the Fourier space formulation of the fluid equations, see LeJan and Sznitman [14] and Bhattacharya et al [1]. The solution obtained in this paper can be written in terms of the Error Function and the heat kernel which have well established probabilistic interpretations in connections with Brownian motion.

References

[1] Rabi N. Bhattacharya, Larry Chen, Scott Dobson, Ronald B. Guenther, Chris Orum, Mina Ossiander, Enrique Thomann, and Edward C. Waymire. Majorizing kernels and stochastic cascades with applications to incompressible Navier-Stokes equations. *Trans. Amer. Math. Soc.*, 355(12):5003–5040, 2003.

[2] Zhi-Min Chen and Tetsuro Miyakawa. Decay properties of weak solutions to a perturbed Navier-Stokes system in $\mathbf{R}^n$. *Adv. Math. Sci. Appl.*, 7(2):741–770, 1997.

[3] David C. Clark. The vorticity at infinity for solutions of the stationary Navier-Stokes equations in exterior domains. *Indiana Univ. Math. J.*, 20:633–654, 1970/1971.

[4] Giuseppe Da Prato and Alessandra Lunardi. On the Ornstein-Uhlenbeck operator in spaces of continuous functions. *J. Funct. Anal.*, 131(1):94–114, 1995.

[5] Robert Finn. Estimates at infinity for stationary solutions of the Navier-Stokes equations. *Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.)*, 3 (51):387–418, 1959.

[6] Robert Finn. On steady-state solutions of the Navier-Stokes partial differential equations. *Arch. Rational Mech. Anal.*, 3:381–396 (1959), 1959.

[7] Robert Finn. On the exterior stationary problem for the Navier-Stokes equations, and associated perturbation problems. *Arch. Rational Mech. Anal.*, 19:363–406, 1965.

[8] Avner Friedman. *Partial differential equations of parabolic type.* Prentice-Hall Inc., Englewood Cliffs, N.J., 1964.

[9] Giovanni P. Galdi. *An introduction to the mathematical theory of the Navier-Stokes equations. Vol. I*, volume 38 of *Springer Tracts in Natural Philosophy.* Springer-Verlag, New York, 1994. Linearized steady problems.

[10] Giovanni P. Galdi. On the motion of a rigid body in a viscous liquid: a mathematical analysis with applications. In *Handbook of mathematical fluid dynamics, Vol. I*, pages 653–791. North-Holland, Amsterdam, 2002.

[11] Giovanni P. Galdi. Steady flow of a navier-stokes fluid around a rotating obstacle. *J. of Elasticity*, 71:1–31, 2003.

[12] Ronald B. Guenther, Robert T. Hudspeth, and Enrique A. Thomann. Hydrodynamic forces on submerged rigid bodies—steady flow. *J. Math. Fluid Mech.*, 4(3):187–202, 2002.

[13] Ronald B. Guenther and John W. Lee. *Partial differential equations of mathematical physics and integral equations.* Dover Publications Inc., Mineola, NY, 1996. Corrected reprint of the 1988 original.

[14] Y. Le Jan and A. S. Sznitman. Stochastic cascades and 3-dimensional Navier-Stokes equations. *Probab. Theory Related Fields*, 109(3):343–366, 1997.

[15] L. Lichtenstein. *Grundlagen der Hydromechanik*. Springer-Verlag, Berlin, 1929.

[16] Wilhelm Magnus, Fritz Oberhettinger, and Raj Pal Soni. *Formulas and theorems for the special functions of mathematical physics*. Third enlarged edition. Die Grundlehren der mathematischen Wissenschaften, Band 52. Springer-Verlag New York, Inc., New York, 1966.

[17] Ryûichi Mizumachi. On the asymptotic behavior of incompressible viscous fluid motions past bodies. *J. Math. Soc. Japan*, 36(3):497–522, 1984.

[18] F.G.K. Odqvist. Über die Randwertaufgaben der Hydrodynamik zäher Flüssigkeiten. *Math. Z.*, 32:329–375, 1930.

[19] F.K.G. Oseen. *Neuere Methoden und Ergebnisse in der Hydrodynamik*. Akademische Velagsgesellschaft, Leipzig, 1927.

[20] Albert H. Schatz, Vidar Thomée, and Wolfgang L. Wendland. *Mathematical theory of finite and boundary element methods*, volume 15 of *DMV Seminar*. Birkhäuser Verlag, Basel, 1990.

[21] Markus Schuster. Computation of the Stresses on a Rigid Body in the Exterior Stokes and Oseen Flows. Master's thesis, Department of Mathematics, Oregon State University, 1999.

[22] Denis Serre. Chute libre d'un solide dans un fluide visqueux incompressible. Existence. *Japan J. Appl. Math.*, 4(1):99–110, 1987.

[23] V. A. Solonnikov. Estimates for solutions of a non-stationary linearized system of Navier-Stokes equations. *Trudy Mat. Inst. Steklov.*, 70:213–317, 1964.

[24] Elias M. Stein. *Singular integrals and differentiability properties of functions.* Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J., 1970.

[25] H. F. Weinberger. Variational properties of steady fall in Stokes flow. *J. Fluid Mech.*, 52(2):321–344, 1972.

[26] H. F. Weinberger. On the steady fall of a body in a Navier-Stokes fluid. In *Partial differential equations (Proc. Sympos. Pure Math., Vol. XXIII, Univ. California, Berkeley, Calif., 1971)*, pages 421–439. Amer. Math. Soc., Providence, R. I., 1973.

[27] G. K. Youngren and A. Acrivos. Stokes flow past a particle of arbitrary shape: a numerical method of solution. *J. Fluid Mech.*, 69:377–403, 1975.

[28] G. K. Youngren and A. Acrivos. Corrigenda: "Stokes flow past a particle of arbitrary shape: a numerical method of solution" (J. Fluid Mech. **69** (1975), 377–403). *J. Fluid Mech.*, 75(4):813, 1976.