ARBITRAGE FREE VALUATION OF A FEDERAL TIMBER LEASE

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Abstract. The objective of this paper is to describe a quantitative framework for offering timber harvest bids on federal lands which includes special considerations of the volatility of timber indices and the harvesting costs involved in harvesting timber. The advantage of such an approach is that it provides a precise framework in which various underlying considerations, such as volatility and cost, may be systematically defined, measured and evaluated. The valuations derived in this paper provide a market standard against which additional value that encompasses social or environmental welfare may be evaluated. We discuss a specific USDA Forest Service timber sale as a case study to illustrate this approach.

1. Introduction

The principle of no-arbitrage provides that two instruments with the same payoff will have the same price. That is, if selling trees or bonds tomorrow will yield the same profit then buying the trees or bonds today should cost the same amount. A timber lease provides the right to harvest the wood subject to a prescribed payment rule, as well as other parameters e.g. duration, base rates, and harvesting costs. In particular this payment rule depends on the type of contract, e.g. escalated or

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non-escalated, and a minimum bid rate, A. This agreed upon rule exposes both the lease writer and holder to a risk of revenue loss or gain depending upon the underlying price of wood at harvest. As the lease writer, the government measures this exposure relative to an underlying weighted average of timber prices, I, referred to as a timber index. The basic idea which will be pursued in this paper is as follows. The government receives a certain amount $g(A)$ for the contract from the lease holder. To offset revenue losses, the government may invest an amount $p(A)$ in a portfolio that matches the exposed revenue loss. Equating $g(A)$ with $p(A)$ leads to an arbitrage free minimum bid rate A appropriate to the terms of the contract and the amount received for the contract. That is the two assets, the timber contract and the off-setting portfolio, yield the same return. The no-arbitrage value for the cutting price A in terms of the contract parameters thus holds these values equal.

To address the issue of obtaining a no-arbitrage value for a federal forest lease we develop a theoretical framework in which to systematically determine a minimal bid rate A at which timber prices should be advertised, given the conditions currently prescribed by Federal contracts.

Of particular interest to us are: (i) The effect of fixed investment costs, and (ii) The problem of incorporating the volatility of timber indices into the determination of A when the risk assumed by the writer is fully diversifiable. The costs of harvesting standing timber are investment costs which, unlike the transaction costs which occur in financial securities markets, are one time costs that the government returns to the lease holder. We shall see that the incorporation of investment costs leads to reasonably simple modifications of the standard no-arbitrage pricing by a Black-Scholes-Merton equation. This is in significant contrast to the difficulties which arise in mathematical pricing theory in the presence of transaction costs; e.g. see Karatzas [7, pp. 117-133].

There are a number of approaches available to incorporate volatility. For example, one may attempt to stochastically model the underlying indices and compute expected present values. However when risk is fully diversifiable such an approach is well-known to lead to prices susceptible to arbitrage in securities markets and, as we shall show in this paper, in timber leases as well. We demonstrate that the no-arbitrage theory of mathematical finance provides a fairness standard (i.e. arbitrage free) from the point of view of the writers of the types of timber lease contracts found in practice.

While we will attempt to make this paper accessible to non-experts in the modern theory and methods of finance, a number of new textbooks have recently appeared which provide more detailed treatments of the essential aspects of the theory required here; e.g. see Dixit and Pindyck [3], Hull [6], and Baxter and Rennie [1]. In particular there are two rather deep results from mathematical finance theory which will play a primary role. One result is that the average rate of return of timber prices is transformed to the risk free interest rate in the final no-arbitrage pricing formula. In particular the numerical value of the average return plays no role beyond formulation of the model, where it is subject to the natural efficient market constraint that the average return be larger than the risk free interest rate; e.g. see Hull [6, pp. 221]. The second standard result is that the no-arbitrage price is a discounted expected value based on a mathematical expectation that involves a special risk-neutral weighting; e.g. see Baxter and Rennie [1, pp. 120]. This new probability weighting, referred to as a martingale change of measure, is explicitly furnished by the theory for the type of timber index model being considered here; namely the geometric Brownian motion model, e.g. see Dixit and Pyndick [3, pp.71], Hull [6, pp.198] or Baxter and Rennie [1, pp.51]. We will explain the essential ideas for these results as they arise in our treatment of timber lease pricing.

In the papers by Morck, Schwartz and Strangeland [9], Reed [12], Reed and Haight [13] for example, the authors seek to compute optimal harvest schedules using geometric Brownian motion models to describe the stochastic evolution of such quantities as prices and timber inventories. While this problem is quite different from that of the present paper, the solutions are obtained under similar hypotheses on the evolution of underlying parameters. It will be of interest to determine time scales on which the statistics of timber indices are captured by this model, however its main justification is its simplicity.

Our interest is in a theoretical determination of the advertised price A based on no arbitrage principles. The fee collected may be viewed as an advertised price plus a premium determined by the auction value. The premium is an independent consideration which is determined by the sealed bid methods. In this regard we call attention to the body of literature available from auction theory for the setting of minimum bids; McAfee and McMillan [8], Paarsch [10]. In particular the authors are grateful to an anonymous referee for pointing out the paper by Paarsch [10] in which it is (empirically) shown how to calculate the minimum acceptable bid for timber sales in order to obtain the desired selling price. This paper complements the auction theory by furnishing a no-arbitrage value of the desired lease selling price.

The overall structure of this paper is as follows: In Section 2 we describe the elements of two particular types of Federal timber leases, escalated and non-escalated. Here standard contract terms are recast in mathematical terms. Since it is not obvious how one may view a timber lease as derivative contract, in Section 3 we provide a brief overview of some basic general principles of mathematical finance which will facilitate the identification of timber leases as certain types of options made in Section 4. Then in Section 4 we introduce a stochastic model for the timber indices and show precisely how the timber leases translate into option contracts. We then calculate the no-arbitrage values of escalated and non-escalated contracts, respectively. From the point of view of standard finance theory the incorporation of costs is a delicate matter and may lead to an incomplete market, see Baxter and Rennie [1, pp.196-200]. For the purpose of hedging the exposed risk, our calculation assumes that the writer has available the same investment opportunities as the bidder. That is, both have access to standing timber. We will see that this leads to a determination of the no-arbitrage value of escalated and non-escalated timber leases. A case study will be provided in Section 5 to numerically illustrate the theory with parameter values used in current practice. Section 6 is a summary of the paper offering discussion and concluding remarks.

2. Elements of a Federal Timber Lease

One of the last steps in timber sale planning is the appraisal which provides the agency's estimate of the value of the timber over the term of the lease. It is the appraised value which is the minimum harvest price (e.g. $\$/m^3$) for which the timber is ultimately advertised to the public for bid. There are two main ways in which the minimum bid is determined in practice, one is by transactions evidence (TE) and the other is by residual value (RV). The TE method begins by estimation of a fair market value (FMV) of stumpage to be sold and then applies a competition premium.

By contrast, RV does not seek a FMV but is the difference of the expected selling value of the end product and an industry average harvesting costs. While the TE method involves sampling error in the average it is not explicitly incorporated into the pricing. On the other hand, in Section 4 we will see that approaches based on expected selling prices, such as RV, are subject to arbitrage regardless of how well the model fits the data; see Baxter and Rennie [1, p.9].

Under the National Forest Management Act, bids on timber leases offered by the Forest Service may be either sealed or oral. However the sample contracts we have analyzed were awarded in competitive sealed bid auctions.

The Forest Service sets an advertised price A which serves as the minimum allowable bid. The maximum bid M is awarded the contract and the awardee is required to post 20% M. This amount (without interest) will be applied to the payment for wood once 25% of the contracted timber amount is harvested. Thus the interest earned $(e^{rt} - 1).2M$, where r is the risk free interest rate and τ is the schedule time by which 25% harvesting occurs, may be viewed as the lease price paid by the awardee for the contract to cut during some specified period of time $[0, T]$ covered by the lease agreement after adjustment to present value, i.e.

$$e^{-rt}(e^{rt} - 1).2M = (1 - e^{-rt}).2M.$$

The lease contract provides an implied obligation to harvest at a price K which is specified net of cost according to two different formulae depending on whether the contract is escalated or not. In an escalated contract the price K depends on the movement of an underlying base price index I. While this index typically lags the actual price of wood [11], it is this index which, after adjustment for cost, provides the government's benchmark price on which decisions are made. In particular, the escalated value of K is prescribed by

$$K = \begin{cases} \max\{B + c, I\} - c = \max\{B, I - c\} & I \leq M + c \\ \frac{I - c + M}{2} & \text{if } I \geq M + c \end{cases} \tag{1}$$

where B denotes a minimum acceptable price that the contract will accept for harvesting, referred to as the base price, c is a cost adjustment, and I is the index value at payment. The base price B and the bid M are both quoted net of cost, whereas the index I is not net of cost. There are two ways in which this can be reconciled in the formulae. Namely, one can either return the costs to the net values or one can make the index I net of cost. As it turns out the former adjustment is mathematically more convenient in the model formulation . The primary reason is that this keeps the index values positive, whereas net of cost indices may take negative values. Moreover, parameters such as the volatility should not depend on the particular operating costs, but rather be determined by the index price evolution. There are a number of different types of cost adjustments which may occur in a specific contract, e.g. average harvesting costs, base rate adjustments, buyer obligated add-ons etc. Note that the specific nature of these costs will be immaterial to our pricing method so long as we are consistent both mathematically and with the contracts we analyze with these adjustments. The point is that while the way in which these costs are distributed will obviously affect the final price, consistency is the only issue for the approach followed here. Finally, by our considerations of auction premiums described above we are interested in the valuation with $M = A$ throughout this paper.

Similar considerations apply to the non-escalated contract where the harvest price, K is prescribed in the lease and is fixed throughout the term of the lease. As we will see in the next section, this is a particularly interesting contract in the way that no-arbitrage considerations give a price which is independent of the stochastic evolution followed by the timber indices. Such a phenomenon is well-known in the pricing of forward contracts; e.g. see Hull [6, p.52], or Baxter and Rennie [1, p.16].

3. Some Principles of Mathematical Finance

The principles of arbitrage theory have their origins in the valuation of financial instruments such as options, forwards, and other derivative contracts. The adaptation of this theory to the valuation of timber leases such as described in the previous section is based on the observation that it is the payoff function to the holder of a contract which essentially defines the "option". Equivalently, the payoff to the holder is an exposure to lost revenue for the writer. For example, a European call option is a contract which gives the holder the "option" to buy a security at some specified strike price K on some date T. Denoting today's value of the security by S_0, the value at a future time t is a stochastic quantity S_t whose evolution is often modeled by a geometric Brownian motion parameterized by an average return μ and a volatility σ ; see Dixit and Pyndick [3, p.71], Hull [6, pp.196-200], Baxter and Rennie [1, p.51].

Arbitrage pricing theory assumes rational decision making to the extent that the call option will be exercised by the holder if and only if the value S_T of the security at maturity T is above the strike price K. Thus the exposure to the writer is a function of the behavior of the underlying asset prices alone. In the present case of a call option it is $S_T - K$ if $S_T > K$, or 0 if S_T K, which is typically expressed more briefly as $(S_T - K)^+$. Mathematically this translates into the working principle that it is the formula for the payoff function in terms of the underlying asset which determines whether or not an "option" is exercised or not. In this way the contract is no more an option than is a forward contract where one is "obligated" to buy a security at some specified strike price K at an agreed upon date T, i.e. a forward is defined by some other payoff function of the underlying prescribed by the rules of the contract, namely $(S_T - K)$ regardless of whether S_T is above or below K. A negative payoff is a loss to the holder.

In view of the above discussion, an option contract is defined by a prescribed payoff function to the holder or an equivalent exposure to lost revenue of the writer. Another example which will be of interest to us here is that of a put contract.

The European put gives the holder the option to sell the underlying security at a specified strike price K and maturity T. Contracts which permit exercise of an option at any time prior to the maturity date T are referred to as American, rather than European. As one might expect, there is a put-call parity relation between the payoffs in that holding a call and writing a put is equivalent to a forward. Examples of more "exotic" contracts may be found in Hull [6, pp. 414-431], Karatzas [7, pp.17-24], for example. Particular exotics which will be observed include barrier options, see Hull [6, p.418]. Here they arise naturally in the context of escalated and non-escalated contracts. Also Asian put and call options, see Hull [6, p. 138], arise naturally in the context of escalated timber leases. Barrier options refer to contracts that are declared void if the underlying price crosses a specified value. On the other hand, Asian options refer to contracts in which the underlying asset or strike price is typically replaced by an average value, e.g. S_T or K replaced by $\int_0^\pi S_\tau \, d\tau$. Such designs discourage artificial manipulation of option values by massive short term speculations on the underlying asset, i.e. hostile takeovers.

As discussed in the introduction the Principle of No $-$Arbitrage is applied to the valuation of financial derivatives via the construction of an off-setting hedging portfolio having the same payoff function as that of the derivative contract. For example, the no-arbitrage price of the European call option defined above is computed as the amount which the writer could invest at time 0 in a portfolio of risk free assets (e.g. US Treasury bonds) plus shares of the underlying asset at its present price S_0 in such a way that suitable management of the portfolio will yield a portfolio value at maturity T which exactly matches the payoff to the holder regardless of the future behavior of the underlying asset prices S_t. Thus one determines the present value of the option to be that amount required to initiate the portfolio investment. This method works to obtain the present price of the option for the two basic reasons that: (i.) There is a market for managing the portfolio of underlying asset and risk free bonds to create a hedge and (ii.) Mathematical methods are available to calculate the initial amount required to create the portfolio used to hedge the writer's exposure. For detailed illustrations for a wide variety of payoff functions see Hull [6], Baxter and Rennie [1], Karatzas [7], for example.

4. Federal Timber Leases as Options: Mathematical Representation

In evaluating the government pricing policy it is assumed that the index I is the standard against which gains and losses are measured by the federal forest management agencies. The arbitrage theory applied here is formulated relative to this index. In particular, the lease is valued from the perspective of the writer, or government. From the lease holder's perspective, while we compute a no-arbitrage price relative to I, arbitrage opportunities may be available in the real timber market since I is not the actual market value for wood.

To formulate a pricing model we must identify the tradable asset and the time horizon of the contract. With regard to the time horizon, we simplify the payment schedule for the contract as follows:

CONDITION E. *Payment for the wood harvested is settled at the expiry T of the lease.*

As a result of this payment schedule the 20% is returned at time T rather than according to some other payment schedule which involves harvesting 25% of the prescribed volume.

We make a final assumption with regard to the development of the off-setting portfolio. Namely we assume that lumber prices track timber values and that standing timber can be used to hedge the risk in the timber contracts. There is currently a futures market for lumber made available by the Chicago Mercantile Exchange.

In order to obtain the government's assessment of a fair price relative to the underlying price index we require a model for the evolution of the underlying price index. For simplicity we assume that the base price index I evolves in time according to a standard stochastic model of the form

$$dI = \mu I \, dt + \sigma I dW(t) \tag{2}$$

where μ is a mean yield rate parameter in the sense that$EI(t) = e^{\mu t} I_0$. The parameter σ defines the volatility in the base price index, and $\{W(t) : t \geq 0\}$ is the standard Wiener's Brownian motion process governing random fluctuations. In addition, while the parameter values $I_0 > c$, T, σ found in our case study make it a rare event, if I happens to fall below c, the contract is void.

Given these assumptions, non-escalated and escalated timber leases may be formulated as option contracts as follows:

Non-escalated Contract. In a non-escalated contract the writer's exposure to lost revenue in terms of the price index I net of cost is simply $V = I - c - A$ for $I > c$ since the contracted price A will be paid regardless of the indices. That is, if $0 < I - c < A$ then the writer has the positive payoff $A + c - I$, but if $I - c > A$, the writer is exposed to a revenue loss in the negative amount $I - c - A$. This is mathematically equivalent to a forward contract with strike price $\tilde{A} = A + c$ and knock-out barrier at c. The knock-out occurs if I falls below c.

Escalated Contract. In an escalated contract the writer's exposure to lost revenue in terms of the cost adjusted price index $I - c > 0$ and the advertised price A is obtained from (1) as

$$V = \begin{cases} I - B - c & c < I \leq B + c \\ 0 & \text{if } B + c \leq I \leq A + c \\ \frac{I-c-A}{2} & I \geq A + c \end{cases} \tag{3}$$

In particular, this is mathematically equivalent to the exposure to lost revenue in holding a put with strike price $\tilde{B} = B + c$ and one half that of writing a call with strike price $\tilde{A} = A + c$, with a knock-out barrier at c. We obtain this revenue loss by observing that: (1) Anytime the value of timber falls below $B + c$, the government will at minimum get B, resulting in a positive payoff, and (2) If the index rises to $I > A + c$, then the lease stipulates that the government receives one-half of the increase from the harvester. That is, the government is only exposed to one-half the lost revenue in a price increase. In particular in the case $\tilde{B} = 0$ this lost revenue V is, up to a factor of one-half, mathematically equivalent to writer's exposure in a European call option.

First let us consider the case of non-escalated contracts. In order to simplify the exposition we will first consider the case of zero costs. In view of the form of the writer's exposure in this case, the contract price $p^{(0)}(A)$ is given by the discounted expected payoff function under the risk-free martingale measure Q and is therefore calculated as

$$p^{(0)}(A) = e^{-rT}E_Q[I - A] = e^{-rT}I_0 - A) \tag{4}$$

The superscript 0 refers to the special case $c = 0$. In particular note that the price (4) is completely independent of the average yield μ and volatility parameter σ of the underlying model (3). Asnoted in the introduction, this is a departure from what one would get by discounting the expected value of wood in (3) to its present value $e^{-rT}e^{\mu T}I_0$. A financial argument which anticipates why no-arbitrage dictates this solution can be made as follows. Suppose that the writer receives an amount v for the contract which is larger than the amount $p^{(0)}(A) = e^{-rT}(e^{rT}I_0 - A) = I_0 - e^{-rT}A$ for the contract. We may assume v is smaller than I_0 for the argument without loss of generality. Then the writer will borrow $I_0 - v$ at the interest rate r and buy the timber to be harvested for the present value I_0. This portfolio will evolve to an amount $I_T - e^{rT}(I_0 - v)$. Now, at expiry he will effectively receive an amount A and give up an intrinsic timber value I_T, leaving a positive wealth $A - e^{rT}I_0 + e^{rT}v > 0$. The inequality is strict, meaning that the amount v is sufficient to repay the loan, cover the exposure and pocket a positive remainder without risk. Thus, a price v above $p^{(0)}(A)$ presents an opportunity for risk free profit when there are no costs. To see that less than this amount is also subject to arbitrage one takes the position of a buyer of such contracts in a similar way.

As discussed at the outset, the amount received by the government for the contract is the interest on the 20% contracted price described in Section 2, so that the no-arbitrage specification of A is the unique positive solution to:

$$(1 - e^{-rT}).2A = p^{(0)}(A) \tag{5}$$

The left hand side of (5) is the interest $(E^{rT} - 1).2A$ earned on the $20\%A$ deposit discounted by e^{-rT} to its present value. Solving (5) using (4) one obtains the non-escalated advertised price as

$$A = \frac{I_0}{.2 + .8e^{-rT}} \tag{6}$$

As is typical for forward contracts, the value of A does not depend on the behavior of the underlying index $I(t)$ beyond its initial value $I(0) = I_0$ at the start of the lease.

Now let us turn to the case of escalated contracts. The basic principles are the same but the writer's exposure depends on the underlying indices in a more complicated manner via (2). First consider the simplest case in which $B = c = 0$. Then the payoff function is equivalent to one-half a call. Thus in this case the no-arbitrage value of the contract in terms of the present timber index I_0 when A is the harvest price is given as a function of A by

$$p^{(0)}(A) = \frac{I_0}{2}\Phi\left(\frac{\left(\log\left(\frac{I_0}{A}\right) + \left(r + \frac{1}{2}\sigma^2\right)T\right)}{\sigma\sqrt{T}}\right) - \frac{A}{2}e^{-rT}\Phi\left(\frac{\left(\log\left(\frac{I_0}{A}\right) + \left(r - \frac{1}{2}\sigma^2\right)T\right)}{\sigma\sqrt{T}}\right) \tag{7}$$

where $\Phi(z) = \frac{1}{\sqrt{2\Pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}x^2}\, dx$ is the standard normal probability distribution function; see Hull [6, p.224] or Baxter and Rennie [1, p.91] for the Black-Scholes-Merton formula for the price of a European call option. Here we use the superscript in $p^{(0)}(A)$ to denote that this is the price in the special case that $B = c = 0$. This is the amount needed to initiate a portfolio of standing timber and risk free bonds which will exactly match the government's exposure when there are no fixed harvesting costs.

The case in which the cost c is positive is a little more delicate. No-arbitrage dictates that the writer has at his disposal the same opportunity to purchase standing timber at $I_0 - c$ as do the perspective holders. In particular, when there are harvesting costs, if the writer's hedge is limited to standing timber at the price I_0 then the cost allowance given to the purchaser of the lease will be insufficient to construct the hedge. This is a departure from standard formulae of finance theory and we must compute the investment required to construct this hedge.

Allowing $c > 0$, and thus $\tilde{B} = B + c \neq 0$, both the escalated and the non-escalated prices may be computed as a solution of the following cost-modified Black-Scholes-Merton equation:

$$\frac{\partial v}{\partial t} + rI\frac{\partial v}{\partial I} + \frac{1}{2}\sigma^2 I^2 \frac{\partial^2 v}{\partial I^2} - rv = cr\frac{\partial v}{\partial I} \tag{8}$$

with boundary condition

$$v(c) = 0 \tag{9}$$

with final values at time T given by (3). However the difference between this and the usual Black-Scholes-Merton equation is in the right hand side of (8); (*cf* equation (10.20) in Hull [6, p.220] or Baxter and Rennie [1, p.95]. A detailed mathematical derivation of (8,9) will be given in a separate paper. To qualitatively understand the appearance of this new term on the right hand side of (8) one must keep in mind that we assume the same opportunities are available to the writer as to the holder when making the hedge. So, while the underlying timber index values I evolve according to the usual model (3), the hedge is based on an off-setting portfolio of standing timber at prices $I - c$ and bonds at the risk free interest rate r.

The cost-modified partial differential equation (8) may be solved as a discounted expected value using the Feynman-Kac formula and standard relationships between stochastic differential equations and parabolic partial differential equations; details are given in Appendix I. Specifically, the solution of (8) with final values at time T given by (3) may be expressed as a discounted expected value of the form:

$$v(I,t) = e^{-r(T-t)} E(V(X_{T-t})) \tag{10}$$

where $X_t,\ t \geq 0$, is the diffusion process defined by the stochastic differential equation

$$dX_t = r(X_t - c)\, dt + \sigma X_t dW_t, \qquad X_0 = I \tag{11}$$

and an absorbing boundary at c.

In the case of a non-escalated contract the price $p^{(c)}(A)$ is obtained from (10) by taking $V = I - A - c$ and $t = 0$. As discussed in the introduction, the advertised price A is obtained by solving

$$(1 - e^{-rT}).2A = p^{(c)}(A) \tag{12}$$

Let us now utilize the specific form of the lease payoff (3) together with some more standard finance theory to express the lease price (10) explicitly in terms of the underlying parameters for an escalated contract. As noted at the outset, the payoff (3) is equivalent to the writer holding a put option with strike price $\tilde{B} = B+c$ and one-half a call with strike price $\tilde{A} = A + c$. Recall that the call and put payoffs are, respectively, given by $V_{\text{call}}(I) = (I - \tilde{A})^+$, and $V_{\text{call}}(I) = (\tilde{B} - I)^+$.

Putting these results together we arrive at the general formula for the price of the escalated lease as

$$p^{(\tilde{B})}(A) = v(I_0, 0) = e^{-rT} EV(X_T) = \frac{1}{2} e^{-rT} EV_{\text{call}}(X_T) + e^{-rT} EV_{\text{put}}(X_T) \quad (13)$$

In particular we obtain the general pricing formula as that of one-half an Asian type call option plus an Asian type put option on an underlying distributed as an arithmetic average of a geometric Brownian motion; see Appendix I(A2) for details. While closed form analytic expressions for such prices are quite complicated, certain mathematical tricks are sometimes used to obtain simple approximations which may be implemented on a programmable calculator; e.g. see Hull [6, p.422]. More comprehensive Monte-Carlo simulation techniques may be found in Gruber [5] and references therein.

With these results we are now equipped to determine the corresponding advertised price A according to the objective stated at the outset of this paper. As above, the fact that the price received for the contract is the interest on the 20% contracted price described in section two yields under Condition E that the fair specification of A is $A = \tilde{A} - c$, where $\tilde{A}$ is the unique positive solution to:

$$(1 - e^{-rT}).2A = p^{(\tilde{B})}(\tilde{A}). \quad (14)$$

Numerical illustrations of solutions to the equation (14) are provided in the next section. However one last calculation needs to be described before we can proceed to the case studies. With the exception of volatility, the parameter values required to determine A may be readily obtained from published resources. To determine the volatility from a time series $I_1, I_2, \ldots, I_{N+1}$ of $N + 1$ sample values of the base price index evolving according to (2) one may take advantage of the form of the solution in the form

$$I(t) = I_0 \exp\left\{\sigma W(t) + \mu t - \frac{1}{2}\sigma^2 t\right\}. \quad (15)$$

In particular the sequence $L_j = \log\left(\frac{I_{(j+1)\delta t}}{I_{j\delta t}}\right)$, $j = 1, 2, \ldots, N$, comprise stationary independent values from a Normal distribution with mean $\left(\mu - \frac{\sigma^2}{2}\right)\delta t$ and variance $\sigma^2 \delta t$. Thus, for the model (2) the minimum variance unbiased estimator of volatility based on this sample of size N is

$$\hat{\sigma}\sqrt{(\delta t)} = \sqrt{\frac{1}{N-1}\sum_{n=1}^{N}(L_j - \overline{L})^2} \quad (16)$$

where $\overline{L} = \frac{1}{N}\sum_{j=1}^{N} L_j$; e.g. see Hull [pp.214-215].

The holder of the timber lease may harvest at any time τ prior to expiry T. This aspect does not effect the result under Condition E and $\tilde{B} = B + c = 0$ since it is well-known for the payoff function of a (non-dividend paying) American call option that there is no advantage to early exercise i.e. the valuation coincides with that of the European call option; see Hull [pp. 158-160], Karatzas [pp. 28-29].

5. Case Study

For a case study we will consider two sample leases furnished to us by the USDA Siuslaw National Forest Service in Philomath, Oregon; one escalated and one non-escalated. These examples furnish a range of realistic parameter values under which we shall calculate escalated and non-escalated prices in this section. As mentioned in the introduction, it is not our intention to reproduce the bid values finally advertised by the forest service, but simply to calculate the corresponding no-arbitrage values based on the information available and under various interest rate, costs and volatility scenarios for comparison. The numerical values are computed with MATLAB and the MATLAB Financial Toolkit.

For our purposes the essential ingredients are first the time series of timber indices available. The monthly Pacific Douglas Fir indices appearing in Table R0800 of the U.S. Forest Service Automated Timber Sales Accounting System, Region 06 Pacific Northwest, are reported in Table 1 for a period covering March,1996 through March, 1997.

Mar 96	Apr 96	May 96	Jun 96	Jul 96	Aug 96	
51.80	54.36	56.67	56.91	60.74	64.32	
Sep 96	Oct 96	Nov 96	Dec 96	Jan 97	Feb 97	Mar 97
63.42	59.92	61.74	60.66	60.55	62.20	61.76

Table 1: Timber Index Values ($\$/m3$; converted)

A plot of this time series is given in Figure 1.

Based on this sample one obtains using (16) an estimate $\hat{\sigma} = .0363\sqrt{12} = .13$, or a volatility of 13%.

The two leases which we consider have appraisal dates of July 9, 1997 and August 18, 1997. The July contract (USDA Forest Service R6-FS-2400-17, Sale Name EXAMPLE) was escalated and the August contract (USDA Forest Service R6-FS-2400-17, Sale Name ID THIN (# 95234) was non-escalated. Both contracts are on Pacific Douglas Fir and prices are quoted in $\$/mbf$ and are converted to metric units ($\$/m^3$) for this paper. The July contract was not sold and was assigned the name EXAMPLE by the Forest Service. The initial values are $I_0 = \$60$ and $I_0 = \$61$, respectively. The complete set of parameter values which we require are summarized in Table 2 (rounded to nearest dollar).

	July Lease EXAMPLE	August Lease ID THIN
I_0	62	60
σ	13%	13%
c	29	13
B	4	2
T		5 yrs
A	24	20
$\tilde{A}$	53	32
$\tilde{B}$	33	15

Table 2: Lease Parameters (costs and values given in $\$/m3$)

The Forest Service priced EXAMPLE as an escalated contract and ID THIN as a non-escalated contract. The cost c refers to stump to truck, haul, road construction and other developments. Also we were not furnished the length of time intended for the lease EXAMPLE, so we assumed it to be $T = 5$ years for the sake of calculations.

In order to compute the advertised price we use a binomial tree to approximate the value of the contract with the cost correction as introduced in (8,9). For a reference on the use of binomial trees to evaluate options, see Hull [pp.335-362]. As in Hull, we choose as binomial tree parameters $u = e^{\sigma\sqrt{\Delta T}}$, $d = \frac{1}{u}$, $p = \frac{(e^{r\Delta T} - d)}{(u-d)}$, and $\Delta T = \frac{1}{52}$ yr. The choice of ΔT corresponds to taking price corrections every week for the duration of the contract. This choice is based on numerical considerations of discretizing a continuous time model.

5.1 Non-Escalated Contract Prices. The non-escalated value computed from (12) and corresponding to the parameter values of ID THIN, $c = 13,\ I_0 = \$61,\ T = 5$yr, with a sample interest rate $r = 5\%$ is, as an illustration, $A = \$58$ (as compared to the Forest Service price of \$20).

Additional solutions to (12) are depicted as points of intersection of the curves in Figure 2 for various cost adjustments $c = \$15, 25, 35, 45$ and contract length $T = 5$yrs. The interest rate is $r = 5\%$.

Note that it is only for the large value of $c = \$45$ that the relation between the advertised price A and the sale price ceases to be linear. In particular for cost values ranging from $c = \$15$ to 35 the advertised price changes in proportion to incremental changes in cost. This relationship does not hold for larger values of c. In fact, in the case of non-escalated sales, the price of the sale can be well approximated by the value of a lease that does not take into account the absorbing boundary condition (9). Moreover, in this case it is possible to solve the differential equation (8) with the final value $v(T, I) = (I - A - c)$ and thus obtain an approximation to the advertised price as

$$A \approx \frac{I_0 - e^{-rT}c}{.2 + .8e^{-rT}} \tag{17}$$

When this approximation is valid, the relation between A and c is linear.

In Figures 3 and 4 we have graphically computed some non-escalated advertised prices A under a variety of different interest rates $r = 1\%$ to $r = 10\%$, and volatilities $\sigma = 1\%$ to $\sigma = 20\%$, respectively.

Once again we note the almost linear relationship between the sale price and the advertised price. Moreover, Figure 4 shows that non-escalated contracts are almost insensitive to changes in the volatility. This observation agrees with the fact that, as noted above, the value of the non-escalated contracts is well approximated by the value given in (17).

In Figure 5 we illustrate the effect of the contract duration on the non-escalated sale. Notice that as the duration of the contract increases so does the value of the contract. This greater value captures the increases in the timber index over time.

5.2. Escalated Contract Prices. The escalated value numerically computed from (14) using the parameter values of EXAMPLE, $c = \$29, I_0 = \$62/m3, T = 5$ yr, $\sigma = 13\%$ and the rate $r = 5\%$ is $A = \$51$ (as compared to the Forest Service price of \$ 24).

Let us remark that one may check for the range of parameter values considered here, that the contribution to the lease value from the "put" term is negligible. To validate this, Figure 6 shows the value of a put price when the base price B, varies from \$ 0 to \$ 50 and $I_0 = \$60, r = 5\%, c = \$25$ and $\sigma = 10\%$.

As we can see from this graph it is only for $B > \$30$ that the effect that the "put" option has on the right hand side of (14) is more than 1. We therefore ignore this term in the calculations that are presented below.

Additional solutions to (14) are depicted as points of intersection of the curves in Figure 7 for the interest rates, $r = 1\%, 5\%, 10\%$. For these illustrations, the following values were used, the initial index $I_0 = \$60$, the cost $c = \$25$, the contract length $T = 5$ yrs and the volatility $\sigma = 10\%$.

We remark that changes in the interest rate have a nonlinear effect on the advertised prices. For example, increasing the interest rate from 5% to 10% only increases the price from \$ 48 to \$ 49. On the other hand, increasing the interest rate from 1% to 5%, causes a drop in price of approximately \$6 per m^3 from \$48 to \$54.

We have graphically computed some escalated advertised prices A under a variety of different cost adjustments in Figure 8. The escalated advertised price under each adjustment is the point of intersection of each set of curves. We kept $I_0 = \$60, r = 5\%, T = 5$yr, and $\sigma = 10\%$.

The figures show that to a good approximation the advertised prices vary linearly with the cost adjustment c, namely an increase in the cost adjustment of \$ 10 corresponds to a decrease in the advertised price of the same amount. We remark however that this is only an approximation since it is also clear from the figure that the curve corresponding to $c = \$15$ is not simply the translation of the curve corresponding to $c = \$5$.

Finally, we also consider escalated pricing as a function of index volatility. Some escalated advertised prices as a function of volatility $\sigma = 1\%, 10\%, 13\%, 20\%$ for $r = 5\%, c = \$25, I_0 = \$60, T = 5$yr are graphically depicted in Figure 9.

The nonlinear dependence of the advertised price on the volatility is apparent from Figure 9. For example, doubling the volatility from 10% to 20% changes the advertised price by approximately \$15. On the other hand, a ten fold increase from 1% to 10% increases the advertised price by approximately \$6.

Comparing the Figures 7,8 and 9, one may conclude that the advertised price is most greatly affected by changes in volatility.

Figure 10 illustrates the effect of the duration on the advertised price. As in the non-escalated case, we see that as the duration of the contract increases, the price does as well. This higher value captures the changes in the index over time.

6. Conclusions and Discussion

In this paper we have provided a forest lease valuation tool that incorporates both price volatility and harvesting costs to determine arbitrage-free minimum bid. One strength of our method is that price changes are incorporated directly into the minimum bid such that the lease writer can cover itself against losses due to price fluctuations. An additional contribution of the paper is that we have developed a method for including harvesting costs in the Black Scholes-Merton option pricing theory.

The case study and corresponding graphs provide illustrations of the effect that interest rates, cost and volatility have on the advertised price. These examples clarify one of the main points of the paper: Volatility can critically affect the lease price and must be systematically included in the pricing structure.

The pricing method of this paper provides a market standard for the value of a timber lease. That is, we may have an objective benchmark against which we may consider valuations that include non-market values such as social and environmental welfare. We have chosen to use arbitrage free methods because they are based on formal mathematical solutions, and sound financial theory. Therefore, we treat the government as a participant in a competitive market in which the necessary trading exists for creating a hedge portfolio.

The reader may notice that our values differ from those that the Forest Service obtained. One reason for this is that in this iteration, we have not included other forest service objectives, such as community payments, or other environmental responsibility. Including these objectives would complicate our presentation of the arbitrage-free method that we seek to introduce with this paper. This observation suggests an interesting extension of this paper. That is, in future work, non-market values such as social and environmental welfare could be modeled within the stochastic, dynamic framework that we present. Thus, our intention here is not to show whether the Forest Service has over or under-valued timber leases, but rather to introduce a valuation procedure based on robust financial methods.

Weaknesses of our approach include the Condition E that the settlement occurs at the end of the contract. An additional caveat is the assumption that we have made regarding the stochastic model for the evolution of indices. Since we have based our model on the index, and not on the actual market, price discrepancy may arise between the writer and the holder (lessee).

Despite these shortcomings this approach introduces a quantitative framework in which policy makers and managers may simply and systematically examine the effects of some key parameter values, in particular interest rates, costs, and volatility, in appraisals of bid prices.

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Appendix

(i) An application of the Cameron-Martin-Girsanov theorem, e.g. see Bhattacharya and Waymire (pp. 616-618, 1990), or Baxter and Rennie (1997), yields a path-dependent representation of the stochastic solution to (11) as for $t < T_c$ as

$$\begin{aligned} X_t &= Ie^{\left\{\sigma W_t + \left(r - \frac{\sigma^2}{2}\right)t\right\}} - rc\int_0^t e^{\left\{\sigma(W_t - W_\tau) + \left(r - \frac{\sigma^2}{2}\right)(t-\tau)\right\}}\,d\tau \\ &= IL_0^{(t)} - rc\int_0^t L_\tau^t\,d\tau \qquad (A1) \end{aligned}$$

where for each $0 \leq \tau \leq t$ the random factors $L_\tau^{(t)}$ are the lognormally distributed variables defined by

$$L_\tau^{(t)} = \exp\left\{\sigma(W_t - W_\tau) + \left(r - \frac{\sigma}{2}\right)(t-\tau)\right\}$$

and T_c is the absorption time for X_t. In particular it is important to observe that the two factors, $L_\tau^{(t)}$ and $\int_0^t L_\tau^{(t)}\,d\tau$ are statistically independent.

(ii) It now follows from (A1) using the Feynman-Kac formula, e.g. see Bhattacharya and Waymire (pp. 606-607), or Baxter and Rennie (1997), that the general formula for the price (13) of the escalated lease is

$$\begin{aligned} p^{(\tilde{B})}(A) = e^{-rT}\frac{1}{2}E&\left\{\left(I_0L_0^{(T)} - rc\int_0^T L_\tau^{(T)}d\tau - \tilde{A}\right)^+ 1(T_c > T)\right\} \\ &+ e^{-rT}E\left\{\left(\tilde{B} - I_0L_0^{(T)} + rc\int_0^T L_\tau^{(T)}\,d\tau\right)^+ 1(T_c > T)\right\} \qquad (A2) \end{aligned}$$