Simulation of Fractional Brownian Surfaces via Spectral Synthesis on Manifolds

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Abstract—Using the spectral decomposition of the Laplace-Beltrami operator we simulate fractal surfaces as random series of eigenfunctions. This approach allows us to generate random fields over smooth manifolds of arbitrary dimension, generalizing previous work with fractional Brownian motion with multidimensional parameter. We give examples of surfaces with and without boundary and discuss implementation.

Index Terms—Fractal surfaces, fractional Brownian motion, discrete Laplace-Beltrami operators.

I. INTRODUCTION

A common approach to simulating fractal surfaces is via the sample paths of fractional Brownian motions and their multidimensional extensions to \( \mathbb{R}^n \) (e.g. [1], [2]). These random fields are self-similar in distribution in that when sampled at various scales the distribution of the sample is the same up to a constant scaling factor. However, they are indexed by \( \mathbb{R}^n \). Suppose instead one wanted to simulate a fractal surface with more complex geometry or topology, e.g., a fractal cylinder. The analogous approach to simulating such an object would require a random field indexed by a manifold, such as a surface in \( \mathbb{R}^3 \), that possessed the same properties as fractional Brownian motion (fBm), i.e., self-similarity, almost surely H"older continuous sample paths, and stationary increments.

Constructing such fields has been a subject of ongoing research for a number of years. Paul Lévy ([3]) was the first to construct over wide classes of surfaces, including arbitrary compact manifolds, both with and without boundary, and for the full range of \( \alpha \in (0,1) \). These extensions were shown to have H"older continuous sample paths, be self-similar in distribution, and have stationary increments, thus possessing all the basic properties one would expect of an extension of fractional Brownian motion. In this paper we describe the simulation of these fields for surfaces in \( \mathbb{R}^3 \).

The motivating mathematics for the technique in this article is a departure from the classical method based on (1). Instead, we take a spectral approach to define our fields. In \( \mathbb{R}^d \), a well-known simulation algorithm for fBm uses Fourier or wavelet techniques (e.g. [1], ch. 2, [2]), but if one is working on a non-Euclidean surface the Fourier transform is no longer available. On the other hand, one can view the Fourier transform as the spectral decomposition of the Laplacian, the sine and cosine functions being the eigenfunctions. These we do have on a surface, and so the analogous approach is to build and simulate random fields over manifolds and surfaces using Fourier series of eigenfunctions of the Laplace-Beltrami operator. One novel feature of this approach is that for a surface with boundary we are able to produce fields with almost sure boundary values. We begin the next section with a derivation in the simple case of the interval \([0,1]\), before moving on to surfaces.

II. FRACTIONAL BROWNIAN FIELDS OVER SURFACES

A. A Basic Example

As motivation and a base case for our construction, consider Brownian motion on \([0,1]\). Being a Gaussian process, \( B_t \) is determined by its covariance function,

\[
\mathbb{E}[B_t B_s] = \min\{s, t\} \equiv s \wedge t.
\]

Now notice that \( s \wedge t \) is the Green’s function of the Laplacian on \([0,1]\), \( -\Delta = -\frac{d^2}{ds^2} \), i.e., for \( f \in L^2[0,1] \)

\[
-\frac{d^2}{ds^2} \int_0^1 s \wedge t f(t) dt = f(s).
\]

Of course we must specify boundary conditions to have a well defined Laplacian, and we see that these come from \( s \wedge t \): If

\[
F(s) = \int_0^1 s \wedge t f(t) dt
\]

then \( F(0) = 0 \) and \( F'(1) = 0 \). Thus we can say that \( B_t \) is the unique (up to equality in distribution) Gaussian process on \([0,1]\) whose covariance is the Green’s function of \(-\Delta \) acting on smooth functions \( f : [0,1] \rightarrow \mathbb{R} \) such that \( f(0) = f'(1) = 0 \).

If we let \( K : L^2[0,1] \rightarrow L^2[0,1] \) be given by

\[
K(f)(s) = \int_0^1 s \wedge t f(t) dt
\]
then we can write (2) as

\[ K = (-\Delta)^{-1} \] (3)

and in this way we see that starting from \( B_t \), we uniquely determine \(-\Delta\) as the inverse of the integral operator \( K \) defined by the covariance of \( B_t \). If \( \{\lambda_k\}_{k=1}^{\infty} \) and \( \{\phi_k\}_{k=1}^{\infty} \) are the eigenvalues and eigenfunctions respectively of \(-\Delta\), with the above boundary conditions, then a calculation shows

\[ \lambda_k = \left( k - \frac{1}{2} \right)^2 \pi^2, \quad \phi_k(x) = \sqrt{2} \sin \left( k - \frac{1}{2} \right) \pi x. \]

The Spectral Theorem and the functional calculus associated with it then yield

\[-\Delta(f) = \sum_{k=1}^{\infty} \lambda_k (f, \phi_k) \phi_k,\]

and

\[ K(f) = \sum_{k=1}^{\infty} \lambda_k^{-1} (f, \phi_k) \phi_k, \]

\((f, g)\) denoting the \(L^2\) inner product, \( \int_0^1 f \, g \, dx \). If we now write

\[ K(f) = \int_0^1 k(x, y) f(y) \, dy, \]

\[ k(x, y) = \sum_{k=1}^{\infty} \lambda_k^{-1} \phi_k(x) \phi_k(y), \]

it is easily seen that the series defining \( k(x, y) \) converges absolutely and uniformly on \([0, 1]\) and moreover it must be that \( k(x, y) = x \wedge y \). Then if \( \{\xi_k\} \) are independent, identically distributed, standard normal random variables, an application of Fubini’s Theorem yields

\[ \mathbb{E} \left[ \left( \sum_{k=1}^{\infty} \lambda_k^{-\frac{1}{2}} \xi_k \phi_k(s) \right) \left( \sum_{k=1}^{\infty} \lambda_k^{-\frac{1}{2}} \xi_k \phi_k(t) \right) \right] = k(s, t) = s \wedge t \]

and we thus arrive at the well known Fourier series expansion of \( B_t \),

\[ B_t = \sum_{k=1}^{\infty} \lambda_k^{-\frac{1}{2}} \xi_k \phi_k(t) = \sqrt{2} \sum_{k=1}^{\infty} \xi_k \frac{\sin \left( k - \frac{1}{2} \right) \pi t}{(k - \frac{1}{2}) \pi}. \] (4)

Now let \( W \) be the white noise (or isonormal process, cf [10]) on \( L^2[0, 1] \) and denote its action a function \( f \in L^2 \) by

\[ W(f) = \int f \, dW. \]

Then \( \{W(\phi_k)\} \) is a set of i.i.d. standard normal random variables, denoted again by \( \{\xi_k\} \), and if

\[ k^{\frac{1}{2}}(x, y) = \sum_{k=1}^{\infty} \lambda_k^{-\frac{1}{2}} \phi_k(x) \phi_k(y) \]

\[ = \sqrt{2} \sum_{k=1}^{\infty} \sin \left( (k - \frac{1}{2}) \pi x \right) \sin \left( (k - \frac{1}{2}) \pi y \right) \]

then

\[ (-\Delta)^{-\frac{1}{2}} f(x) = \int_0^1 k^{\frac{1}{2}}(x, y) f(y) \, dy \]

and

\[ (-\Delta)^{-\frac{1}{2}} (W) = \int k^{\frac{1}{2}}(x, y) dW(y) \]

\[ = \sum_{k=1}^{\infty} \lambda_k^{-\frac{1}{2}} \int \phi_k(y) dW \phi_k(x) \]

\[ = \sum_{k=1}^{\infty} \xi_k \sin \left( (k - \frac{1}{2}) \pi t \right). \]

Thus we can write \( B_t = (-\Delta)^{-\frac{1}{2}} (W) \) and arrive at the stochastic steady-state equation

\[ (-\Delta)^{-\frac{1}{2}} B_t = W. \] (5)

Notice that working backwards we can define \( B_t \) to be the unique Gaussian process satisfying (4). This is similar, but not the same as, the classical equation

\[ \frac{d}{dt} B_t = W. \]

To see that (4) is in fact different, note that \((-\Delta)^{-\frac{1}{2}}\) is self adjoint on \( L^2[0, 1] \), while \( d/dt \) is not (integrate by parts).

Now suppose we replace the mixed Laplacian above by the Dirichlet Laplacian in (5), acting on functions such that \( f(0) = f(1) = 0 \). The Dirichlet Laplacian on \([0, 1]\) has eigenvalues and eigenfunctions given by

\[ \lambda_k = (k \pi)^2 \]

and

\[ \sqrt{2} \sin(k \pi x) \]

respectively. Then we have as above a process \( X_t \) defined by

\[ X_t = \sum_{k=1}^{\infty} \xi_k \frac{\sin(k \pi t)}{k \pi}. \]

If we now let \( \alpha \in (0, 1) \) (the case above being \( \alpha = 1/2 \)) we can extend (5) to

\[ (-\Delta)^{\left( \frac{1}{2} + \alpha \right)} X_t = W \] (6)

and write

\[ X_t = \sum_{k=1}^{\infty} \xi_k \frac{\sin(k \pi t)}{(k \pi)^{\left( \frac{1}{2} + \alpha \right)}}. \] (7)
which is the homogeneous Riesz field constructed in [9]. This is a self-similar Gaussian process with almost surely H"older continuous sample paths and such that \( X_0 = X_1 = 0 \) almost surely, i.e., it is a Gaussian bridge.

Because the series converges in \( L^2 \) with probability 1, we can simulate these processes by taking partial sums (see Figure 1). Moreover, if \( X_i^N \) denotes the \( N \)-th partial sum in (7) then by definition we have

\[
E[X_i^N X_s^N] = \sum_{k=1}^{N} \frac{\sin(k\pi t) \sin(k\pi s)}{(k\pi)^{1+2\alpha}}, \tag{8}
\]

which converges absolutely and uniformly, i.e., the field \( X_i^N \) converges to \( X_t \) in distribution on \([0, 1]\) (cf [10]).

Notice that as \( \alpha \) increases the sample paths become more regular. The reason can be seen from (7): Lower alpha emphasizes the larger eigenvalues and thus the higher frequency wave functions show more of a contribution. Similarly a larger \( \alpha \) suppresses their contribution and the lower frequency wave functions dominate, leading to less overall oscillation and a smoother sample path. We will see this same behavior in general below.

B. Extension to Cylinders

Suppose now \( D \) is a compact connected surface in \( \mathbb{R}^3 \) with smooth boundary. Then there is a canonical differential operator on \( C^\infty(D) \) corresponding to the Euclidean Laplace operator above, the Laplace-Beltrami operator, which we refer to also as simple the Laplacian of \( D \) (see e.g. [11]). As above we let \( \lambda_k \) and \( \phi_k \) be the eigenvalues and eigenfunctions of the Dirichlet Laplacian of \( D \), \(-\Delta\). Given a white noise \( W \) on \( D \) we start from

\[
(-\Delta)^{\frac{\gamma}{2} + \frac{\alpha}{2}} R_x = W
\]

to obtain

\[
R_x = \sum_{k=1}^{\infty} \lambda_k^{-\frac{\gamma}{2} - \frac{\alpha}{2}} \xi_k \phi_k(x), \tag{9}
\]

where the change in the exponent from 1/4 to 1/2 ensures sample path regularity (this is required by the change in dimension, see [9] and section IV below). In this way we obtain a random field on \( D \) that is self-similar, almost-surely H"older continuous of order \( \gamma \) for any \( \gamma < \alpha \) and almost surely not H"older continuous for \( \gamma \geq \alpha \), and has stationary increments. A few words are in order as to what we mean by self-similar: Because we are considering surfaces embedded in \( \mathbb{R}^3 \), that is, surfaces with Riemannian metrics induced by the Euclidean metric on \( \mathbb{R}^3 \), the usual definitions carry over, i.e., \( R_x \) is not defined. We can instead consider the series

\[
R_x^\alpha \overset{d}{=} \sum_{k=1}^{\infty} (\lambda_k)^{-\frac{\alpha}{2} - \frac{\alpha}{2}} (\phi_k(x) - \phi_k(o)) \xi_k \tag{10}
\]

where \( o \) is some fixed point serving as an “origin.” This series does converge and again defines a self-similar field with stationary increments and almost sure H"older continuity as above, called the Riesz Field in [9]. Figure 3 shows spheres boundary circles, i.e., the height of the field values plotted along normals is zero.

C. Spheres

For surfaces without boundary there are no obvious boundary conditions, so we need to interpret our equations differently. The basic issue is that on a compact manifold, e.g., the sphere \( S^2 \), the lowest eigenvalue of the Laplacian is zero and \((-\Delta)^{\frac{\gamma}{2} + \frac{\alpha}{2}}\) is not defined. We can instead consider the series

\[
R_x^\alpha \overset{d}{=} \sum_{k=1}^{\infty} (\lambda_k)^{-\frac{\alpha}{2} - \frac{\alpha}{2}} (\phi_k(x) - \phi_k(o)) \xi_k
\]

Fig. 2. Cylinders with Dirichlet boundary conditions for \( \alpha = .1, .5, .9 \), \( N = 200 \).
with increasing alpha. Here the spectrum is again known analytically, the eigenfunctions being the spherical harmonics, allowing us to plot the partial sum approximation as above.

III. OTHER MANIFOLDS

Although the above examples have not been constructed before, and are thus of interest even for the relatively simple case of the sphere, one of course may often be required to work with a more general underlying surface. For most manifolds analytical expressions for the eigenvalues and eigenfunctions of the Laplacian are not available, and so one must resort to some approximation scheme. The discretization of the Laplace-Beltrami operator is the subject of active and ongoing research, and though activity in the field continues to quicken, results giving conditions for eigenstructure convergence (for example see condition SC below) of a discretized operator are still incomplete. While there is no shortage of operators to choose from ([12], [13]), much of the theory regarding spectral approximation remains to be developed. Here we simply give a sufficient condition for the convergence properties required of the discrete Laplace-Beltrami operator for accurate simulation of our fields.

Suppose we have a manifold $M$ and have chosen a discrete Laplacian, $L_h$ with corresponding eigenvectors and eigenvalues $\{\phi^h_k\}$ and $\{\lambda^h_k\}$, $h$ denoting some parameter, for example the longest edge length of an approximating mesh. Let $R$ denote one of the Riesz fields above, and $\bar{R}$ the field generated using $L_h$.

A simple condition for the approximation in distribution $\bar{R} \xrightarrow{d} R$ is as follows: Suppose that for any integer $N > 0$, it holds that for each $n \leq N$

$$\lim_{h \to 0} \| \phi^h_n - \phi_n \|_{\infty} = 0$$

and

$$|\lambda^h_n - \lambda_n| \to 0,$$

where we can embed $\phi^h_n$ in $C(M)$. We refer to the above as the spectral convergence condition, SC.

If SC holds then letting $\bar{R}^N$ denote the field given by

$$\sum_{n=1}^{N} (\lambda^h_n)^{-\frac{1}{2}} \phi^h_n(x) \xi_n,$$

we have

$$\lim_{h \to 0} \left\| \sum_{n=1}^{N} (\lambda^h_n)^{-\frac{1}{2}} \phi^h_n(x) \xi_n - \sum_{n=1}^{N} \lambda_n^{-\frac{1}{2}} \phi_n(x) \xi_n \right\|_{\infty} = 0,$$

e.g., $\bar{R}^N$ converges to $R^N$ in distribution as $h \to 0$. As remarked above, as $N \to \infty$ $R^N$ converges to $R$ in distribution, so one need only choose $N$ sufficiently large and then take $h$ toward zero to approximate $R$ by $\bar{R}^N$ in distribution.

Unfortunately, conditions like SC are currently only known to hold in limited cases. In his dissertation, Wardetzky demonstrates that the well-known cotangent scheme satisfies SC (see [14]). However, it requires certain conditions on the mesh and in [15] the authors show that higher order FEM easily outperforms the cotangent operator in accuracy. Interestingly, Belkin and Niyogi have shown that spectral convergence holds for the graph Laplacian when the vertices are sampled at random from a uniform density on the manifold. See Dey [14] and the references therein. Perhaps most promising, an operator was introduced in [16] and shown to give pointwise convergence, and in [14] convergence of eigenvalues was demonstrated along with experimental evidence for convergence of eigenspaces.

Once one has a discrete operator satisfying SC, one can simulate fractional brownian fields on arbitrary compact manifolds. In our previous examples the eigenfunctions are explicitly known in closed form, guaranteeing an accurate simulation. However, for the special case of compact planar domains, [17] shows that the 5-point Laplacian does yield eigenstructure convergence satisfying condition SC. This allows us to simulate some novel planar surfaces with complex boundaries. See Figure 4 for an example of a fractional Brownian field plotted on an asymmetric domain for various $\alpha$. 

![Spheres with different alpha values](image-url)
A. Further Remarks on Convergence and Approximation

There are two parameters which determine the computation time in simulating \( R_x \) for a given surface: The parameter \( h \) controlling the convergence of the discrete Laplacian \( L_h \), e.g., how fine the mesh is, and the number \( N \) of eigenvectors appearing in

\[
R_x = \sum_{k=1}^{N} (\lambda_k)^{-\left(\frac{\alpha}{2} + \frac{\alpha}{2}\right)} \xi_k \tilde{\varphi}_k(x).
\]

If we are working with one of the above examples where we use known analytical formulas for the eigenfunctions then of course all that matters is \( N \).

Recall that the series defining \( R_x \) converges in \( L^2 \) almost surely. Weyl’s asymptotic formula for the eigenvalues of \( \Delta \) (e.g. [11]) tells us that \( \lambda_k = O(k) \) for a two dimensional surface as we are dealing with here. Thus

\[
\lambda_k^{-\left(\frac{\alpha}{2} + \frac{\alpha}{2}\right)} = O\left(k^{-\left(\frac{\alpha}{2} + \frac{\alpha}{2}\right)}\right)
\]

and so we see that the contribution, in mean square, of the \( N \)th term is of the order \( N^{-\left(\frac{\alpha}{2} + \frac{\alpha}{2}\right)} \). We see here how \( \alpha \) affects the quality of the approximation. Once \( N \) is chosen, one must choose \( h \) depending on the discrete Laplacian being used and with the available theory as a guide.

The computation of even the first \( N \) eigenvectors and eigenvalues of \( L_h \) can become long for a given \( h \). However, once computed, this spectral data can be stored for repeated use. All that is needed is the random numbers \( \{\xi_k\} \), which are not expensive. Moreover, the self-similarity of the Riesz fields allows one to compute the spectra of a surface of any size, and simply rescale the resulting field without computing the spectra for a rescaled surface. For example, once one has simulated the Riesz field on a sphere of radius \( c \) simply by plotting \( c^n R_x \) for the appropriate value of \( \alpha \). In this way, for very detailed images, the computational time required may be somewhat high, but it is a one time cost in the above sense.

IV. Possible Extensions

We have restricted our attention to embedded surfaces in \( \mathbb{R}^3 \), however the theory in [9] applies to non-embedded manifolds, e.g., hyperbolic space or flat tori, as well as \( n \)-dimensional surfaces in \( \mathbb{R}^{n+1} \). In this case the approximation takes the form

\[
R_x = \sum_{k=1}^{N} (\lambda_k)^{-\left(\frac{\alpha}{2} + \frac{\alpha}{2}\right)} \xi_k \tilde{\varphi}_k(x).
\]

Of course working with a surface whose metric is not induced by the ambient Euclidean metric can increase the difficulty in computing accurate discretizations of the Laplacian. For example, the discrete Laplacian of [16] is explicitly defined in terms of an ambient Euclidean geometry. Also, we have here chosen \( \alpha \) to be fixed, but one could also allow \( \alpha \) to vary over the surface and thus obtain multifractal fields. We expect, similar to what is known on \( \mathbb{R}^n \), that such fields with varying fractal index would still be continuous.

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