Comparative Statics: Classical and Modern Approaches*

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If you want literal realism, look at the world around you; if you want understanding, look at theories. (Dorfman, 1964)

We would, of course, dismiss the rigorous proof as being superfluous: if a theorem is geometrically obvious why prove it? This was exactly the attitude taken in the eighteenth century. The result, in the nineteenth century, was chaos and confusion: for intuition, unsupported by logic, habitually assumes that everything is much nicer behaved than it really is. (Steward, 1975)

I am pleased that the seemingly endless disputes on the role of mathematics in economics have largely ceased.... Without rigor, the author and the reader simply cannot evaluate whether a result is right or wrong. (Allen, 2000)

*This is required reading for Econ 513, spring 2007. In spite of being over 15 years old, these modern methods are missing from undergraduate and graduate textbooks in mathematical economics. This is a rough draft. Comments are welcome, but please do not quote without the author’s permission. (File Name: Econ-513 – Comparative Statics – 2007).
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1. Introduction

Comparative statics or sensitivity analysis investigates how the endogenous variables of a model are affected by a change in a parameter or exogenous variable. The “comparative” term refers to a before and after comparison of an optimum or equilibrium value that results from a very small change in an exogenous variable or parameter. The “statics” term refers to the fact that a comparison is made after all adjustments have occurred. That is, the dynamic process of going from one outcome to another is ignored. Such before and after comparisons provide testable predictions and policy implications of economic models.

Comparative static analysis is performed on equilibrium and optimization models. The classic approach applies the implicit-function theorem to first-order conditions in optimization models and to equilibrium conditions in equilibrium models.1 To apply the implicit-function theorem, however, certain regularity conditions must hold. For example, derivatives of relevant functions must be continuous, objective functions must be concave, and the equilibrium must be stable. An important weakness of the implicit-function theorem is that it is not applicable for discrete changes in parameters or exogenous variables.

Recent work in the area of monotone comparative statics demonstrates, however, that comparative static analysis can be done without many of the restrictions required by the implicit-function theorem.2 For example, this approach works for discrete as well as infinitesimally small changes in parameters or exogenous variables. The “monotone” term refers to statements about order; that is, an increase in one variable leads to an increase in another variable. Of course, something is given up with this new approach. Unlike the implicit-function theorem, monotone comparative statics can tell us the direction but not the magnitude of change.

Another concern is that comparative static analysis becomes more difficult to apply in game theoretic settings. With many players and choice variables, the curse of dimensionality is a problem. That is, it becomes increasingly tedious (algebra intensive) to calculate the result. In addition, multi-equilibria are common in many games, making it difficult to apply the implicit-function theorem. Recent work shows, however, that unambiguous comparative static results can emerge when a game exhibits qualities of super-complementarity. Such a game is said to be supermodular and exists when each player’s own choice variables are complementary and all

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1For a review of the this approach, see Simon and Blume (1994), Hand (2004), Baldani et al. (2005), and Chiang and Wainwright (2005). For a discussion of alternatives, such as the revealed preference approach, see Carter (2001).

strategic variables across players are strategic complements.\footnote{For a discussion of strategic complements and supermodular games, see Bulow et al. (1985), Milgrom and Roberts (1990) and Vives (1999, 2005a, and 2005b).}

This note is written with three goals in mind. The first is to review comparative statics using the implicit-function theorem. The second is to show how monotone comparative static methods can be used when there are discrete changes. The final goal is to show how comparative static analysis can be performed in games that are supermodular.

2. Comparative Statics and Implicit Functions

One of the most important professional activities of economists is to carry out exercises in comparative statics: to estimate the consequences and merits of change in economic policy and our economic environment. (Scarf, 1994, 111)

In economics, most comparative static problems involve answering the following question: How does a change in a parameter (or exogenous variable) affect the equilibrium or optimal value of an endogenous variable in a model? Although tedious, this is relatively easy when the structural equations of a model take on a specific functional form and can be solved explicitly for the optimal value of an endogenous variable.

To illustrate, consider a simple demand and supply model with the following structural equations:

\[
Q_D = a - bP + cY \\
Q_S = eP
\]

where \(Q_D\) is quantity demanded, \(Q_S\) is quantity supplied, \(P\) is price, and \(Y\) is consumer income. Price and quantity are endogenous variables, and income is exogenous. The parameters \(a, b, c,\) and \(e\) of the are assumed to be positive. That is, demand has a negative slope, the commodity in this model is a normal good, and supply has a positive slope. To determine the effect of income on the equilibrium price (\(P^*\)), we can solve explicitly for \(P^*\) and differentiate the function with respect to income. In equilibrium, \(Q_D = Q_S\) and the equilibrium price is

\[
P^* = \frac{a + cY}{b + e}.
\]

In this case, the effect of income on the equilibrium price is

\[
\frac{dP^*}{dY} = \frac{c}{b + e},
\]

which is positive, given the assumptions of the model. That is, the model predicts that an increase in income will lead to an increase in the equilibrium price of a normal good.

In more complex models, this approach may be impractical or impossible to use. For example, it may be difficult to solve explicitly for the equilibrium price, which can occur when demand and supply are non-linear. More importantly, how do we perform comparative static
In three dimensions, this neighborhood literally includes an open ball. With two choice variables, it is an open disk. With one choice variable, it is an open interval. In the case of a single choice variable, for example, the neighborhood around point $x^* = 2$ might be: $1 < x < 3$, written as $x \in (1, 3)$.

**Implicit-Function Theorem:** Let $F(x, y)$ be a function with partial derivatives that exist and are continuous in a neighborhood (called an open ball $B$) around the point $(x_1^*, ..., x_n^*, y^*)$, such that $(x_1^*, ..., x_n^*, y^*)$ satisfies:

$$F(x_1^*, ..., x_n^*, y^*) - c \equiv 0;$$

$$\frac{\partial F(x_1^*, ..., x_n^*, y^*)}{\partial y} \neq 0.$$

Then $F(x, y) - c$ uniquely implies a function $y = f(x_1, ..., x_n)$ that has derivatives that are continuous within the open ball $B$ around point $(x_1^*, ..., x_n^*)$ such that:

A. $F(x_1, ..., x_n, y(x_1, ..., x_n)) = c$ for all $(x_1, ..., x_n)$ within $B$,  
B. $y^* = f(x_1^*, ..., x_n^*)$, and  
C. for each $i = 1, ..., n$,

$$\frac{\partial y(x_1^*, ..., x_n^*)}{\partial x_i} = -\frac{\frac{\partial F(x_1^*, ..., x_n^*)}{\partial x_i}}{\frac{\partial F(x_1^*, ..., x_n^*)}{\partial y}}$$

The result in part C is sometimes called the **implicit-function rule**.
Proof of the Implicit-Function Rule: Take the total differential of \( F(x_1, \ldots, x_n, y(x_1, \ldots, x_n)) = c \) around the point \( (x_1^*, \ldots, x_n^*, y^*) \),

\[
\frac{\partial F}{\partial x_1} dx_1 + \cdots + \frac{\partial F}{\partial x_n} dx_n + \frac{\partial F}{\partial y} dy = 0
\]

Note that the asterisks are suppressed for convenience. Assuming we are interested in \( \frac{\partial y}{\partial x_1} \), set \( dx_2 = dx_3 = \ldots = dx_n = 0 \). Solving for \( \frac{dy}{dx_1} \),

\[
\frac{dy}{dx_1} \bigg|_{x_2,x_3,\ldots,x_n} = -\frac{\partial F}{\partial x_1} \bigg/ \frac{\partial F}{\partial y}.
\]

Note by definition that this equals \( \frac{\partial y}{\partial x_1} \) because we have set \( dx_2 = dx_3 = \ldots = dx_n = 0 \). Q.E.D.

Part C of the implicit-function theorem can determine the magnitude of change when specific functions are used to describe the structural model. With general functional forms, we are only able to determine the direction of change. Several examples illustrate this approach.

2.1 Comparative Statics in Equilibrium or Optimization Models

The most common economic applications of the implicit-function theorem are to equilibrium models and models that involve optimization techniques. Several examples are illustrated below.

Example 1 (Equilibrium Problem): Consider the demand and supply problem described above with the following excess demand function: \( ED(P, Y) = Q_d(P, Y) - Q_s(P) \). Let the partial derivatives of the excess demand function be continuous, the demand function have a negative slope (i.e., \( \frac{\partial Q_d}{\partial P} < 0 \)), the commodity be a normal good (i.e., \( \frac{\partial Q_d}{\partial Y} > 0 \)), and supply function have a positive slope (i.e., \( \frac{\partial Q_s}{\partial P} > 0 \)). As before, our goal is to determine how an increase in \( Y \) will affect the equilibrium price. Assuming an interior equilibrium exists, the excess demand function will be identically equal to zero at equilibrium values of price and quantity. Under these conditions, the implicit function implies the following:

\[
\frac{dP}{dY} = -\frac{\frac{\partial ED}{\partial Y}}{\frac{\partial ED}{\partial P}} = -\frac{\frac{\partial Q_d}{\partial Y}}{\frac{\partial Q_d}{\partial P} - \frac{\partial Q_s}{\partial P}} > 0.
\]

\[\text{For a proof of the implicit-function theorem, see Lang (1983).}\]

\[\text{The assumption of an open ball is required to rule out corner solutions. With two choice variables, this is an open disk. With one choice variable, this is an open interval.}\]
Note that stability of the demand and supply model requires that $\frac{\partial ED}{\partial P} < 0$. When this condition holds, price rises when there is excess demand ($ED > 0$), causing excess demand to fall; price falls when there is excess supply ($ED < 0$), causing excess supply to fall. Stability conditions are commonly used to derive comparative static results in equilibrium problems.

**Example 2 (Optimization Problem):** Consider a monopoly firm that faces a per-unit tax ($t$), and our goal is to determine the effect of the tax on the firm’s optimal (profit-maximizing) output level. The firm’s profit is: $\pi(q, t) = TR(q) - TC(q) - tq$, where $\pi$ is profit, $q$ is output, $TR$ is total revenue, and $TC$ is total production cost. Assuming a differentiable and strictly concave profit equation, the respective first- and second order-conditions of profit maximization are:

$$\frac{\partial \pi}{\partial q} = \frac{\partial TR}{\partial q} - \frac{\partial TC}{\partial q} - t = 0;$$

$$\frac{\partial^2 \pi}{\partial q^2} = \frac{\partial^2 TR}{\partial q^2} - \frac{\partial^2 TC}{\partial q^2} < 0.$$

Although it cannot be derived explicitly, embedded in the first-order condition is the profit maximizing level of output. At this optimal value, the first-order condition is identically equal to zero and the implicit-function theorem implies that

$$\frac{dq^*}{dt} = -\frac{\frac{\partial^2 \pi}{\partial q^2}}{-\frac{\partial^2 TR}{\partial q^2} + \frac{\partial^2 TC}{\partial q^2}}.$$

Because the denominator is negative from the second order condition, this derivative is negative. That is, an increase in an excise tax will reduce a monopolist’s profit maximizing output level. This example shows how the second-order condition is used to derive comparative static results in optimization problems.

**Example 3 (Optimization Problem):** Consider a problem where we want to determine how an increase in the price of an input will affect a firm’s profit maximizing quantity of that input (i.e., the slope of the firm’s profit-maximizing input demand). Assume the firm is a price taker (i.e., both input and output markets are perfectly competitive) and uses just two inputs: labor ($L$) and capital ($K$). The firm’s profit is: $\pi = pq(L, K) - wL - rK$, where $w$ is the price of labor and $r$ is the rental rate of capital. Assume the problem is short run, where capital is fixed and labor is variable (the long-run problem will be considered next). Assuming a differentiable and strictly concave profit equation, the respective first and second order conditions of profit maximization are:

$$\frac{\partial \pi}{\partial L} = p \frac{\partial q}{\partial L} - w = 0;$$

$$\frac{\partial^2 \pi}{\partial L^2} = p \frac{\partial^2 q}{\partial L^2} < 0.$$

Embedded in the first-order condition is the optimal value of labor, and from the implicit-function theorem:
Again, this example illustrates how the second-order condition plays a key role in the comparative static analysis in optimization problems. When the second-order condition is met, the marginal product of labor will have a negative slope and the firm’s profit maximizing demand for labor will have a negative slope.

Most optimization problems in economics have more than one choice variable. For example, General Motors uses varying degrees of skilled to unskilled labor, different types of raw materials (e.g., steel, plastic, leather), and different types of physical capital to produce cars, trucks, and refrigerators. In cases such as these, first-order conditions become a system of linear equations that must be solved simultaneously to find the optimal values of the choice variables. When these functions are implicit and include many variables, it becomes convenient to differentiate the first-order conditions and solve the resulting system using matrix algebra and Cramer’s rule (Simon and Blume, 1994, p. 194). To illustrate, consider the previous problem, but now more than one input is variable.

**Example 4 (Optimization Problem with a System of Equations):** Assume that a perfectly competitive firm wants to maximize its long-run profit with respect to all of its inputs. For \( n \) inputs, the profit equation is

\[
\pi = pq(x) - \sum_{i=1}^{n} w_i x_i,
\]

where \( x_i \) is the quantity of input \( i \), \( w_i \) is the corresponding input price, and \( x \) is a vector of \( n \) inputs. For simplicity, let \( n = 2 \). The system of first-order conditions is

\[
\begin{align*}
\pi_1 &= pq_1 - w_1 = 0, \\
\pi_2 &= pq_2 - w_2 = 0.
\end{align*}
\]

For notational convenience, let \( \pi_i \) equal the first derivative of the profit with respect to input \( i \) and \( q_i \) equal the first derivative of the production function with respect to input \( i \). Because \( p \) and \( w \) are exogenous, \( \partial \pi/\partial x_i/\partial x_j = \partial q/\partial x_i/\partial x_j \) for all \( i \) and \( j \) equal to 1 and 2. Again, for convenience, we write this as \( q_{ij} \). Thus, the matrix of second derivatives of the profit equation, the Hessian matrix \( (H) \), can be written as

\[
H = \begin{pmatrix}
q_{11} & q_{12} \\
q_{21} & q_{22}
\end{pmatrix}
\]

For a unique maximum, the profit (production) equation must be strictly concave. For this to occur, the Hessian matrix is negative definite (second-order conditions): \( q_{ii} \) must be negative and the determinant of the Hessian matrix must be positive. In this problem, our goal is to determine how a change in \( w_i \) will affect \( x_i^* \) and \( x_j^* \). By substituting the optimal input values into the first-order conditions, making them identically equal to zero, we can differentiate each first-order condition with respect to \( w_i \). This yields the system of equations:
This is a system of two linear equation in two unknowns \((\partial x_i^*/\partial w_i)\) and \((\partial x_j^*/\partial w_j)\), which can be written in matrix form as

\[
\begin{pmatrix}
pq_{ii} & pq_{ij} \\
pq_{ji} & pq_{jj}
\end{pmatrix}
\begin{pmatrix}
\frac{\partial x_i^*}{\partial w_i} \\
\frac{\partial x_j^*}{\partial w_j}
\end{pmatrix}
\equiv
\begin{pmatrix}
1 \\
0
\end{pmatrix}
\]

Applying Cramer’s rule to this system produces the comparative static results of interest

\[
\frac{\partial x_i^*}{\partial w_i} = \frac{1}{|H|} \begin{vmatrix}
pq_{ii} & pq_{ij} \\
pq_{ji} & pq_{jj}
\end{vmatrix} = \frac{pq_{ii}}{p(q_i q_j - q_i^*)},
\]

\[
\frac{\partial x_j^*}{\partial w_j} = \frac{1}{|H|} \begin{vmatrix}
pq_{ii} & pq_{ij} \\
pq_{ji} & pq_{jj}
\end{vmatrix} = \frac{-pq_{ji}}{p(q_i q_j - q_j^*)}.
\]

As in previous optimization problems, the denominators in both results are strictly positive by the second-order conditions. Because second-order conditions also require \(q_i\) to be negative, the own price effect will always be negative. That is, the long-run demand for all inputs will have a negative slope. The cross price effect is indeterminate, however. If the two inputs are complements in production \((q_{ij} > 0)\), then \(\partial x_j^*/\partial w_j < 0\); if the two inputs are substitutes in production \((q_{ij} < 0)\), then \(\partial x_j^*/\partial w_j > 0\).

Comparative static analysis of the objective function itself is also important in economics and involves the use of the **envelope theorem**. This says that if we plug in the optimal values of the choice variables into the objective function, the derivative of this function (frequently called the value function or indirect objective function) with respect to a parameter is equal the direct effect of the parameter on the original objective function. When applied to a firm’s profit function, this is called Hotelling’s lemma, which is illustrated in the following example.

**Example 5 (Optimization Problem and the Envelope Theorem):** Let us continue with the example above. Optimal input values, \(x_i^*\) and \(x_j^*\), derive from the system of first-order conditions. Plugging them into profit equation produces the profit function (or indirect objective function) is
\[ \pi^*(w_i, w_j, p) = pq(x_i^*, x_j^*) - w_i x_i^* - w_j x_j^*. \]

To determine how maximum profit changes with a change in \(w_i\), we can differentiate the profit function,

\[
\frac{\partial \pi^*}{\partial w_i} = p \left( q_i \frac{\partial x_i^*}{\partial w_i} + q_j \frac{\partial x_j^*}{\partial w_i} \right) - w_j \frac{\partial x_j^*}{\partial w_i} - x_i^* - w_j \frac{\partial x_j^*}{\partial w_i},
\]

\[ = \left( pq_i - w_j \right) \frac{\partial x_i^*}{\partial w_i} + \left( pq_j - w_j \right) \frac{\partial x_j^*}{\partial w_i} - x_i^*. \]

Because the terms in parentheses are the first-order conditions and equal zero at the optimum,

\[ \frac{\partial \pi^*}{\partial w_i} = -x_i^*, \]

which equals \(\partial \pi / \partial w_i\) when evaluated at \(x_i^*\). This means that the marginal effect on profit of a change in \(w_i\) has only a direct effect. That is, there is no indirect effect through the optimal values of inputs. It also implies that the negative of the demand function for input \(i\) equals the derivative of the profit function with respect to input \(i\). This provides a proof of the envelope theorem for the special case of a profit function, Hotelling’s lemma.

### 2.2 Comparative Statics in Problems with Both Equilibrium and Optimization

The preceding section discussed comparative static analysis for either an equilibrium or an optimization problem. Solutions to game theoretic problems, however, typically involve solutions to both maximization and equilibrium problems. The most widely used equilibrium concept in non-cooperative game theory is the Nash equilibrium, and it will be our focus here. The first applications of the Nash equilibrium concept to firm behavior are the Cournot (1838) and Bertrand (1883) models of duopoly. In the Cournot model, the choice variable is output; in Bertrand, it is price. Because price is more common than output competition, we focus on the Bertrand model.

**Example 6 (Problem with Both Equilibrium and Optimization):** In a differentiated Bertrand model, two firms (1 and 2) produce differentiated products in a single market and compete by simultaneously choosing price. The demand function for firm \(i\) (1 or 2) is \(q_i(p_i, p_j)\), where \(q_i\) is the firm’s output, \(p_i\) is the firm’s price, and \(p_j\) is the price of the firm’s rival. Let demand have a negative slope \(\frac{\partial q_i}{\partial p_i} < 0\) and, because they are substitute products, \(\frac{\partial q_i}{\partial p_j} > 0\). Firm interdependence is revealed in the demand functions; an increase in the price of one firm raises the other firm’s demand, *ceteris paribus*. The total revenue of firm \(i\) is \(TR_i(p_i, p_j) = p_i q_i(p_i, p_j)\). Let total cost for firm \(i\) \([TC_i(q_i, a)]\) increases with its own output and parameter \(a\), where \(a\) is a cost parameter such that \(\frac{\partial TC_i}{\partial a} > 0\) and \(\frac{\partial TC_i}{\partial q_i}/\partial a > 0\). For example, \(a\) could be the price of an important input or a per-unit tax. Our goal is to determine how a change in \(a\) will affect Nash equilibrium prices \((p_i^*, p_j^*)\).

Given these definitions, profit for firm \(i\) is \(TR_i(p_i, p_j) - TC_i(q_i(p_i, p_j), a)\). Note that an increase in \(a\) will raise a firm’s marginal returns of raising price for a negatively sloped demand
function. That is, $\partial (\partial \pi / \partial p_i) / \partial a = \pi_{ii} > 0$. Respective first- and second-order conditions of profit maximization for firm $i$ are:

$$\frac{\partial \pi}{\partial p_i} = \frac{\partial TR}{\partial p_i} - \frac{\partial TC}{\partial p_i} = 0;$$

$$\frac{\partial^2 \pi}{\partial p_i^2} = \frac{\partial^2 TR}{\partial p_i^2} - \frac{\partial^2 TC}{\partial p_i^2} < 0.$$ Note that subscripts are suppressed in profit, total revenue, and total cost.

The best reply function for firm $i$ is determined by solving the firm’s first-order condition for $p_i$: $p_i^{BR}(p_j) = p_i^*$. This identifies the optimal $p_i$ for all values of $p_j$. The optimal value of $p_i$ is embedded in firm $i$’s first-order condition, which is identically equal to zero at $p_i^*$. Thus, we can apply the implicit-function theorem to firm $i$’s first-order condition to determine the slope of its best-reply function. This is

$$\frac{dp_i^*}{dp_j} = \frac{-\pi_{ii}}{\pi_{ia}}.$$

As in previous examples, $\pi_{ii}$ is defined as the second derivative of firm $i$’s profit with respect to $p_i$ and $p_j$, and $\pi_{ii}$ is the second derivative of firm $i$’s profit function with respect to $p_i$. The slope of the best reply will be positive, because prices are strategic complements (i.e., $\pi_{ij} > 0$) as defined by Bulow et al. (1985) and because $\pi_{ii} < 0$ from the second-order condition of profit maximization. Thus, best reply functions in a Bertrand game will have a positive slope.

We determine the effect of an increase in $a$ on Nash prices as follows. By substituting the optimal prices into the first-order conditions of each firm, we can differentiate them with respect to $a$. This yields the system of equations:

$$\pi_{11} \frac{\partial p_1}{\partial a} + \pi_{12} \frac{\partial p_1}{\partial a} + \pi_{1a} \frac{\partial a}{\partial a} = 0,$$

$$\pi_{21} \frac{\partial p_2}{\partial a} + \pi_{22} \frac{\partial p_2}{\partial a} + \pi_{2a} \frac{\partial a}{\partial a} = 0.$$  

This linear system can be written in matrix form as

$$\begin{pmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{pmatrix} \begin{pmatrix} \frac{\partial p_1}{\partial a} \\ \frac{\partial p_2}{\partial a} \end{pmatrix} = \begin{pmatrix} -\pi_{1a} \\ -\pi_{2a} \end{pmatrix}.$$  

Applying Cramer’s rule,

$$\frac{\partial p_i^*}{\partial a} = \frac{\begin{vmatrix} -\pi_{1a} & \pi_{12} \\ -\pi_{2a} & \pi_{22} \end{vmatrix}}{\Pi},$$

where $\Pi$ is the 2 x 2 matrix of second derivatives of profits from the matrix form of the previous linear system above. To have a stable Nash equilibrium, the determinant of $\Pi$ must be positive (i.e., $\pi_{11} \pi_{22} - \pi_{12} \pi_{21} > 0$). This is proven in Appendix A. In addition, second order conditions
require that $\pi_{ii} < 0$. Because prices are strategic substitutes, $\pi_{ij} > 0$. Given these and the fact that $\pi_{ia} > 0$, an increase in $a$ will cause $p_1^*$ to increase. Because the problem is symmetric, $p_2^*$ will also increase with $a$. This example shows how both stability conditions and second-order conditions are important to comparative static analysis in problems that involve both optimization and equilibrium concepts.

3. Monotone Comparative Statics

To apply classic comparative static methods, certain assumptions are necessary. These typically include concavity of the objective function, convexity of constraint sets, and smoothness of objective functions and constraints. Recent work in monotone comparative statics demonstrates that many comparative static conclusions can be obtained with weaker assumptions (Milgrom and Shannon, 1994, Shannon, 1994, and Edlin, Shannon, 1998, and Quah, 2007).

For example, an important weakness of the implicit-function theorem is that it requires differentiability. This is particularly problematic in policy analysis, where a particular policy has a discrete character. For example, an occupational safety regulation is either in effect or not. Likewise, a pollution abatement policy may completely ban the use of certain inputs. Both types of regulations may cause a discrete jump in a firm’s objective function, making it impossible to use the implicit-function theorem. The tools needed to analyze such problems are monotone methods, which show that differentiability and convexity are not always required to perform comparative static analysis. The application of this approach is simple when there is just one choice variable, but higher order problems require a knowledge of lattice theory. For this reason, the illustrations below focus on choice problems in just two dimensions.

To compare classic and new methods, assume that an economic agent wants to maximize an objective function, $f(x, a)$ with respect to $x$, where $a$ is a continuous policy variable. If an interior solution exists and the second derivative of $f$ is continuous, then, from the implicit-function theorem,

$$\frac{dx^*}{da} = -\frac{\partial^2 f / \partial x \partial a}{\partial^2 f / \partial x^2}.$$  

From the second order condition, $\partial^2 f / \partial x^2 < 0$. Thus, the effect of $a$ on $x^*$ will be positive (negative) when $\partial^2 f / \partial x \partial a$ is positive (negative). This method is invalid, however, for a discrete change in the policy variable (e.g., when the discrete variable takes on a value of 0 or 1). To do comparative static analysis in this case, we must use Edlin and Shannon’s (1998) strict monotonicity theorem. To distinguish the continuous from the discrete policy variable, let $a$ equal $\alpha$ in the discrete case.

**Strict Monotonicity Theorem:** Let $f: \mathbb{R} \rightarrow \mathbb{R}$, $S \subset \mathbb{R}$, $x^* = \arg\max_{x \in S} f(x, \alpha^*)$, and $x^* = \arg\max_{x \in S} f(x, \alpha^*)$. Suppose that $x^*$ is a unique interior solution and that $df/\partial x$ is continuous and has strictly increasing marginal returns with respect to the parameter $\alpha$. Then $x^*>x$ if $\alpha^*$

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7For a review of lattice theory, see Milgrom and Shannon (1994) and Topkis (1998).
Similarly, $x^* > x'$ if $\alpha^* > \alpha'$.

Note that in the continuous case, strictly increasing marginal returns means that the parameter and the choice variable are complements. That is, $\partial f/\partial x \partial \alpha > 0$. 

Proof: The proof of the strict monotonicity theorem hinges on the assumption of strictly increasing marginal returns, which means that $\partial f/\partial x (x^*, \alpha)$ is increasing in $\alpha$. Given this definition, the basic argument is as follows. Because $x^*$ is the unique argmax at $\alpha^*$, $f(x, \alpha')$ must increase as we move from $\alpha'$ to $\alpha^*$. It must be true that $x^* > x'$, because if $x^* < x'$, then strictly increasing marginal returns implies that increasing $x$ from $x^*$ to $x'$ must lead to an increase in $f(x, \alpha^*)$. But this contradicts the definition that $x^*$ is the optimal choice at $\alpha^*$. Q.E.D.

To illustrate the intuition behind the theorem, consider a specific functional form. Suppose the objective function is $f(x, \alpha) = g(\alpha)x - x^2$. The parameter $\alpha$ can take on two discrete values, such that $g(\alpha') = 2$ and $g(\alpha^*) = 3$. These two objective functions are illustrated in Figure 1a and are labeled $f'$ and $f^*$. The function exhibits strictly increasing marginal returns in $\alpha$ because the slope of the tangent to the objective function increases as $\alpha$ increases from $\alpha'$ to $\alpha^*$. In other words, $\partial f'/\partial x > \partial f'/\partial x$ as shown in Figure 1b. Thus, by the strict monotonicity theorem, the argmax of $f(x, \alpha)$ increases from $x'$ to $x^*$ as $\alpha$ increases from $\alpha'$ to $\alpha^*$. This example highlights the role of first-order conditions and illustrates how to apply the theorem – essentially all that needs to be checked is whether or not the function exhibits strictly increasing marginal returns with respect to the parameter in question.

Example 7 (Optimization Problem with Discrete Policy Change): Consider a monopoly problem where government policy ($\alpha$) causes a discrete decrease in the cost of doing business. This would include a discrete reduction in an excise tax or a policy that causes a discrete cut in bureaucratic red tape that is imposed on the firm. Our goal is to determine how this policy will affect the firm’s optimal (profit-maximizing) level of output. The firm’s profit is: $\pi(q, \alpha) = TR(q) - TC(q, \alpha)$. Profit is assumed have a derivative in $q$ that is continuous, and an increase in $\alpha$ from $\alpha'$ to $\alpha^*$ causes a discrete reduction in total and marginal cost. Assume further that and $q'$ is the unique argmax of $\pi(q, \alpha')$ and that $q^*$ is the unique argmax of $\pi(q, \alpha^*)$. Under these conditions, the profit equation exhibits increasing marginal returns as $\alpha$ increases from $\alpha'$ to $\alpha^*$. This is demonstrated below.
\[
\frac{\partial \pi(\alpha')}{\partial q} - \frac{\partial \pi(\alpha)}{\partial q} > 0;
\]
\[
\left( \frac{\partial TR}{\partial q} - \frac{\partial TC(\alpha')}{\partial q} \right) - \left( \frac{\partial TR}{\partial q} - \frac{\partial TC(\alpha)}{\partial q} \right) > 0;
\]
\[
- \frac{\partial TC(\alpha')}{\partial q} + \frac{\partial TC(\alpha)}{\partial q} > 0,
\]
which holds because the marginal cost under regime \( \alpha' \) is greater than the marginal cost under regime \( \alpha \) by definition. Thus, by the strict monotonicity theorem, the profit-maximizing level of output increase with a government deregulation that reduces marginal cost.

It is also easy to see from examples 2 and 8 that the implicit-function theorem is a special case of the strict monotonicity theorem. When the policy parameter \( \alpha = a \) is continuous, as in a per-unit subsidy (or excise tax reduction), for example, there will be strictly increasing marginal returns to \( a \). In the continuous case, this implies that \( \partial \pi/\partial q/\partial a > 0 \). From the second-order condition, \( \partial^2 \pi/\partial q^2 < 0 \). Thus, the strict monotonicity theorem implies that a marginal increase in this parameter will cause an increase in the firm’s profit maximizing output level. By the implicit-function theorem, these conditions imply the same result.

A weaker version of the theorem applies when the objective function is neither smooth nor concave (Milgrom and Shannon, 1994). One example is provided in Figures 2a and 2b. In this case, \( f(x, \alpha) \) exhibits strictly increasing differences in \( \alpha \), which is a discrete version of strictly increasing marginal returns. That is, \( f(x, \alpha) \) has strictly increasing differences in \( x \) and \( \alpha \) when for all \( x' > x \), \( f(x', \alpha) - f(x, \alpha) \) is increasing in \( \alpha \). Another example, this time where the objective function is not concave, can be seen in Figure 3. Under the weaker conditions of these two examples, however, the Milgrom and Shannon (1994) theorem states that the optimal value of \( x \) will be non-decreasing in \( \alpha \). This can be seen in the example in Figure 4 where \( \alpha \) has no effect on the optimal value of \( x \) even though the objective function exhibits increasing differences in \( \alpha \). When \( f(x, \alpha) \) is not differentiable in \( x \), this weaker monotonicity theorem implies that \( \partial x^*/\partial \alpha \geq 0 \). This example illustrates why differentiability is required for the strict monotonicity theorem to hold.

The problem is more complicated when there are multiple choice variables. In the single choice problem, checking for increasing marginal returns or increasing differences is essentially all that is needed to do monotone comparative static analysis. With multiple choice variables, the problem is more complicated due to possible interact effects. In this case, monotonicity theorems also requires that all choice variables be complementary. In the continuous case where the objective function is \( f(x_1, x_2, \alpha) \), complementarity of choice variables means that \( \partial f/\partial x_1/\partial x_2 \geq \)
When this conditions holds for all choice variables, the objective function is said to be **supermodular**. Thus, the application of monotone comparative static analysis when there are multiple choice variables requires that all parameters and choice variables be complementary. That is, one must check for both supermodularity and increasing marginal returns (or increasing differences). The concept of supermodularity will be especially important in the next section involving comparative static analysis in game theoretic settings.

### 4. Monotone Methods and Game Theory

This class [of supermodular games] turns out to encompass many of the most important economic applications of noncooperative game theory (Milgrom and Roberts, 1990, 1255).

In a game theoretic setting, both optimization and equilibrium concepts are required to find the solution. This issue was illustrated in Example 6 above, where two firms completed by simultaneously choosing price (i.e., Bertrand duopoly). Each firm was a profit maximizer and the equilibrium concept was Nash. With two firms, there was a first-order condition for each firm, and comparative static analysis required us to differentiate each first-order condition with respect to the parameter in question and to use Cramer’s rule to solve the resulting system of equations. This approach suffers from the so called *curse of dimensionality*, as finding the solution to such games becomes excessively tedious as the number of firms and the number of choice variables (e.g., price, advertising, product quality) increase. In addition, this technique cannot be used when the relevant functions are not differentiable.

Fortunately, comparative static analysis can still be derived using monotone methods. What is required is for the game to be supermodular. Because the main ideas are the same in the differentiable and non-differentiable cases, we will focus on problems where the curse of dimensionality is at issue, not differentiability. These are called **smooth supermodular games**. To illustrate, assume an industry has $n$ firms and each firm has two strategic variables: price ($p$) and marketing expenditures ($M$). Each firm’s profit depends on its own and rival prices and marketing. Firms compete in a smooth supermodular game, defined below.

**Definition:** Firms play a smooth supermodular game if the following conditions hold for each firm $i$ and each rival $j$ (Milgrom and Roberts, 1990, p. 1264).

(A1) **Bounds on Strategies:** $p_i$ and $M_i$ each lie within a closed interval where \( \{p_i \mid 0 < p_{il} \leq p_i \leq p_{ih}\} \) and \( \{M_i \mid 0 < M_{il} \leq M_i \leq M_{ih}\} \).

(A2) **Differentiability:** The profit equation is twice continuously differentiable with respect to $p_i$ and $M_i$.

(A3) **Complementary Strategies:** \( \frac{\partial^2 \pi_i}{\partial p_i \partial M_i} \geq 0 \).

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10This implies that best-reply functions are differentiable, but one could assume more generally that best-replies are complete lattices instead of smooth functions without affecting the main conclusions. For further discussion, see Milgrom and Roberts (1990) and Vives (1999).
A strict inequality will hold if the best-reply functions in prices have a positive slope and if an increase in advertising shifts the best-reply functions away from the origin. Best-reply functions will generally have a positive slope in a differentiated Bertrand-type game where firms compete in prices (Bulow et al., 1985, and Tirole, 1988, chapter 5). As will be seen shortly, advertising can shift best-reply functions toward or away from the origin, causing prices to decrease or increase.

(A4) Strategic Complements: $\frac{\partial^2 \pi_i}{\partial p_i \partial p_j} \geq 0$, $\frac{\partial^2 \pi_i}{\partial p_i \partial M_j} \geq 0$, $\frac{\partial^2 \pi_i}{\partial M_i \partial p_j} \geq 0$, and $\frac{\partial^2 \pi_i}{\partial M_i \partial M_j} \geq 0$.

Assume further that a policy parameter, $a$, has the following qualities: $\frac{\partial^2 \pi_i}{\partial p_i \partial a} \geq 0$ and $\frac{\partial^2 \pi_i}{\partial M_i \partial a} \geq 0$. That is, each firm’s profit equation exhibits increasing marginal returns in price and marketing for an increase in $a$. This assumption is identified as (A5). Such a policy could include an excise tax that raises the marginal returns to raising price or a government subsidy to marketing that raises the marginal returns in marketing. Under these conditions and assuming a unique Nash equilibrium, the following comparative static results hold for all firms (Milgrom and Roberts, 1990, Theorem 6):

$$\frac{\partial p_i^*}{\partial a} \geq 0; \quad \frac{\partial M_i^*}{\partial a} \geq 0.$$  

That is, an excise tax that increases prices will have a non-negative effect on the Nash equilibrium level of marketing, and a subsidy that increases marketing will have a non-negative effect on Nash equilibrium prices in this supermodular setting.  

The proof of this result requires the use of lattices, so it will not be presented here. The main idea is intuitive, however. The driving force behind the proof in the case of a strict inequality is the super-complementarity assumption. That is, any policy that increases $p_i^*$ ($M_i^*$), causes $M_i^*$ ($p_i^*$) to rise because own choice variables are complements (assumption A3), and it also causes $p_j^*$ and $M_j^*$ to rise for all $j$ because rival choice variables are strategic complements (assumption A4). This causes a chain of events that reinforces these increases. That is, the resulting increases in $p_j^*$ and $M_j^*$ cause further increases in $p_i^*$ and $M_i^*$ etc. In terms of best-reply functions, this means that the policy change causes one or both of the best reply functions for each choice variable to shift away from the origin (e.g., from equilibrium A to B in Figure 5 for choice variable $x$). Thus, the Nash equilibrium, where the best reply functions intersect, will support higher levels of the strategic variables.

The following example illustrates three comparative static approaches to an extension of Example 6 above for linear demand and cost functions.

**Example 8 (Example 6 with Specific Functional Forms):** In a differentiated Bertrand model, two firms (1 and 2) face linear demand and costs. Firm $i$’s respective demand and costs are: $q_i = a - b p_i + d q_j$ and $TC_i = (c - t) q_i$, where $a > c > 0$ and $b > 2d > 0$. Policy Parameter $t$ represents a per-unit tax on the firm. Thus, the firm $i$’s profit equals: $\pi = (p_i - c - t)(a - b p_i + d p_j)$. In this case, the respective first and second derivatives of the profit equation are:

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11A strict inequality will hold if the best-reply functions in prices have a positive slope and if an increase in advertising shifts the best-reply functions away from the origin. Best-reply functions will generally have a positive slope in a differentiated Bertrand-type game where firms compete in prices (Bulow et al., 1985, and Tirole, 1988, chapter 5). As will be seen shortly, advertising can shift best-reply functions toward or away from the origin, causing prices to decrease or increase.
\[ \pi_i = a - 2bp_i + dp_j + b(c + t), \]
\[ \pi_{ij} = \pi_{ji} = -2b, \]
\[ \pi_{ij} = \pi_{ji} = d, \]
\[ \pi_{it} = b. \]

With this notation, \( \pi_i = \partial \pi/\partial p_i, \pi_{ij} = \partial^2 \pi/\partial p_i \partial p_j, \pi_{ij} = \partial^2 \pi/\partial p_i \partial p_j, \pi_{it} = \partial^2 \pi/\partial p_i \partial t \) for firm \( i \).

To compare approaches, each of the three techniques discussed in this paper is used to determine the effect of the excise tax on Nash equilibrium prices.

**1) Brute Force Method:** Because specific functional forms are used, we can solve the system of first-order conditions for \( p_1 \) and \( p_2 \) and differentiate. Because the problem is symmetric, the Nash equilibrium prices will be the same for each firm and equal

\[ p_1^* = \frac{a + b(c + t)}{2d - b}. \]

As a result, \( \partial p_i^*/\partial t = b/(2d - b) > 0 \), because \( 2d - b \) is greater than 0 by definition.

**2) Implicit-Function Rule and Cramer’s Rule:** From Example 6, we have seen that

\[ \frac{\partial p_i^*}{\partial \alpha} = \left| \begin{array}{cc} \pi_{\alpha} & \pi_{\gamma} \\ \pi_{\alpha} & \pi_{\delta} \end{array} \right| / |\Pi| \]

The determinant of \( \Pi = 4b^2 - d^2 \) and must be positive for stability. This condition holds, because \( b > 2d \) by assumption. The numerator in the above equation equals \( -\pi_{\alpha} \pi_{\gamma} + \pi_{\delta} \pi_{\gamma} = -(b)(-2d) + (b)(d) > 0 \). Thus, \( \partial p_i^*/\partial t > 0 \).

**3) Supermodularity Theorem:** Alternatively, because the game is supermodular, \( \partial p_i^*/\partial t > 0 \) by the supermodularity theorem. The supermodularity and complementarity parameter assumptions (A1-A5) are required for the theorem to hold and are verified below.

A1. This condition is met as long as \( p_i \in [0, \infty) \).
A2. The second derivatives of \( \pi \) are continuous.
A3. This assumption is not relevant, because each firm has only one choice variable.
A4. \( \pi_{ij} = d > 0 \), verifying that prices are strategic complements.
A5. \( \pi_{ij} = b > 0 \), the excise tax rate is a complementary exogenous parameter.

A strict inequality holds in this case, because an excise tax causes the best reply functions of both firms to shift away from the origin and shift the Nash equilibrium from point A to point B in Figure 6.

This example demonstrate how easy it is to use the supermodularity theorem.
5. Conclusion

In this note, I have reviewed the implicit-function theorem and its limitations. I also present recent work using monotone methods. The strict monotonicity theorem provides a more general method of doing comparative static analysis. When the assumptions required of the implicit-function theorem are met, I have also shown that the implicit-function theorem is a special case of the strict monotonicity theorem. Finally, I have shown how tedious and difficult it can be to derive comparative static results from games with many players and strategic choices. Fortunately, when the game is supermodular, the supermodularity theorem demonstrates that comparative static analysis can be done with ease. I hope that his convinces you to learn these modern methods and use them in future research.
Appendix A: Stability of the Bertrand-Nash Equilibrium

Consistent with the duopoly model in section 2.1, assume smooth best reply functions and an unique interior Nash equilibrium. The graph of the best reply functions assumes that $p_1$ is on the vertical axis and $p_2$ on the horizontal axis. Stability requires that for any disequilibrium set of prices in the neighborhood of the Nash equilibrium, the dynamic process of each firm responding to its rival’s disequilibrium price converges to the Nash prices. This will occur when firm 2’s best reply function is steeper than firm 1’s best reply function. You should verify this fact by starting from a disequilibrium point.

As demonstrated above, best reply functions for firm’s 1 ($p_1^* = b_1$) and 2 ($p_2^* = b_2$) will have a positive slope. Recall that they are

$$b_1 = \frac{-\pi_{12}}{\pi_{11}}; \quad b_2 = \frac{-\pi_{21}}{\pi_{22}}.$$  

Because we are interested in solving each best reply function for $p_1$ (i.e., $p_1$ is on the vertical axis), the slope firm 2’s best reply when $p_1$ on the vertical axis is $1/b_2$. Thus, the Bertrand-Nash equilibrium will be stable iff $1/b_2 > b_1$. Thus:

$$\frac{-\pi_{22}}{\pi_{21}} > \frac{-\pi_{12}}{\pi_{11}}$$

$$\frac{\pi_{22}}{\pi_{21}} < \frac{\pi_{12}}{\pi_{11}}.$$  

Because $\pi_{21} > 0$ and $\pi_{11} < 0$, this becomes

$$\pi_{11} \pi_{22} > \pi_{12} \pi_{21}$$

$$\pi_{11} \pi_{22} - \pi_{12} \pi_{21} > 0.$$  

Q.E.D.
References


Figure 2a

Figure 2b
Figure 3

Figure 4