Chapter 9 Quantity vs. Price Competition in Static Oligopoly Models

We have seen how price and output are determined in perfectly competitive and monopoly markets. Most markets are oligopolistic, however, where more than one but less than many firms compete for consumer business. Firms face a strategic setting in oligopoly markets, because firm profits and, therefore, the best course of action depend critically upon the behavior of competitors as well as on basic demand and cost conditions. Thus, the appropriate method for analyzing an oligopoly setting is game theory.

In this chapter, we focus on the static models of Cournot and Bertrand, models that were developed long before modern game theoretic methods. The choice variable is output in the Cournot model, while the choice variable is price in the Bertrand model. We will see that these models produce different examples of a Nash equilibrium. Recall that this is different from the monopoly case, where the firm's optimal price-output pair is the same whether the firm uses output or price as its strategic variable.

For simplicity, we initially assume homogeneous goods and a duopoly market where just two firms compete against one another. This minimizes mathematical complication but still allows us to analyze many of the important features of firm strategy. We also extend the model to n firms and cost asymmetries. Once these simple models with homogeneous goods are mastered, we analyze duopoly markets with product differentiation. We then turn to the firm's choice of output or price competition and consider issues of strategic substitutes and complements. But first we present the basic duopoly model with homogeneous products.

9.1 Cournot and Bertrand Models with Homogeneous Products

9.1.1 The Cournot Model with Two Firms and Symmetric Costs

The first formal model of duopoly was developed by Augustin Cournot (1838). He describes a market where there are two springs of water that are owned by different individuals. The owners sell water independently in a given period. He also assumed that the owners face zero costs of production and that consumer demand is negatively sloped. As in previous chapters, we assume a linear inverse demand function, $p = a - bQ$, where $Q = q_1 + q_2$ and the parameters a and b are positive.[1] Each owner then sets output to maximize its profits, and the equilibrium price clears the market (p^*). The Cournot problem is to determine the optimal values of our variables of interest: q_1, q_2, p_1, and p_2. Notice that because the products are homogeneous, however, that $p_1 = p_2 = p^*$ in equilibrium if both firms are to participate.

Recall that for this to be a game, we must define the players, their choice variables, their payoffs, and the information set. For the remainder of this chapter, we will assume that information is complete (i.e, payoffs and the characteristics of the game are common knowledge). Because we are also interested in studying how costs affect the market, we will assume that costs are positive and the firm i's total cost equation is $TC_i = cq_i$, $c \geq 0$. In terms of notation, c is the unit cost of production, subscript i signifies firm 1 or 2, and subscript j signifies the other firm. Each firm's goal is to choose the level of output that maximizes profits, given the output of the other firm. The relevant characteristics of the game are:

- Players: Firms or owners 1 and 2.

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[1] Dixit (1979) shows that this demand system derives from the utility function $U(q_1, q_2) = \alpha_1 q_1 + \alpha_2 q_2 - (\beta_1 q_1^2 + \beta_2 q_2^2 + 2\gamma q_1 q_2)/2$, where $a = \alpha_1 = \alpha_2$, $b = \beta_1 = \beta_2 = \gamma$.

- Strategic Variable: Firm i chooses non-negative values of q_i.
- Payoffs: Firm i's payoffs are profits, $\pi_i(q_i, q_j) = pq_i - cq_i = (p - c)q_i = [a - b(q_i + q_j)]q_i - cq_i = aq_i - bq_i^2 - bq_iq_j - cq_i$.

The Nash equilibrium solution to this game turns out to be the same as the Cournot solution. For this reason, it is sometimes called the Cournot-Nash equilibrium or the Nash equilibrium in output in a duopoly game.

At the Nash equilibrium, recall that each firm must behave optimally assuming that its rival behaves optimally. That is, firm i maximizes profits, believing that firm j maximizes its profits. Another way of saying this is that each firm calculates its best response or reply to the expected best-reply behavior of the other firm. For this reason, the Nash equilibrium is sometimes described as a mutual best reply. Mathematically, this means that firm i's profit-maximizing problem takes the following form: max $\pi_i(q_i, q_j)$ with respect to q_i, assuming that firm j chooses its Nash equilibrium output level. The first-order conditions are[2]

$$\frac{\partial \pi_1}{\partial q_1} = a - 2bq_1 - bq_2 - c = 0, \tag{9.1}$$

$$\frac{\partial \pi_2}{\partial q_2} = a - bq_1 - 2bq_2 - c = 0. \tag{9.2}$$

Embedded in each firm's first-order condition is the optimal value of its choice variable. Thus, solving equations (9.1) and (9.2) simultaneously for q_1 and q_2 produces the Nash equilibrium

$$q_1^* = q_2^* = \frac{a - c}{3b}. \tag{9.3}$$

Given the market demand and firm profit functions, the Nash equilibrium price and profits are

$$p^* = \frac{(a + 2c)}{3}, \tag{9.4}$$

$$\pi_1^* = \pi_2^* = \frac{(a - c)^2}{9b}. \tag{9.5}$$

It is easy to verify that the Cournot output and profits levels increase with a decrease in marginal cost and an increase in demand (i.e., as a increases and b decreases). Price rises with an increase in marginal cost and the demand intercept. These comparative static results are the same as those found in the monopoly model with linear demand and cost functions.

An important feature of this mode is that it is symmetric. You can see by inspecting the first order conditions of both firms. Notice that they are similar – we can obtain firm 2's first-order condition by replacing subscript 1 with 2 and subscript 2 in firm 1's first-order condition. Similarly, we can derive firm 1's first-order condition from firm 2's first-order condition. This *interchangeability condition* leads to a symmetric outcome where Nash equilibrium strategies are the same. Symmetry will typically occur when firms have the same cost functions, produce homogeneous goods, and have the same goals. But models may be symmetric when costs differ and products are differentiated as well, issues we will take up later in the chapter.

Because the Cournot outcome is so important to theoretical work in industrial organization, we also want to describe it graphically. First, we describe the equilibrium in terms of best-reply functions. Best-reply functions are obtained by solving each firm's first-order condition for q_2, allowing rival output to vary. Next, we graph these functions, with q_2 on the vertical axis and q_1 on the horizontal axis. They are called best-reply (or sometimes best-

[2] Notice that the second-order condition of profit maximization holds, because the second derivative of the profit function for each firm is $-2b < 0$.

response or reaction) functions because each function describes firm i's optimal output for any value of q_j. From equations (9.1) and (9.2), the best-reply functions for firm 1 (BR_1) and firm 2 (BR_2) are

$$BR_1: q_2 = \frac{a-c}{b} - 2q_1, \tag{9.6}$$

$$BR_2: q_2 = \frac{a-c}{2b} - \frac{1}{2}q_1, \tag{9.7}$$

Notice that these functions are linear and expressed in slope-intercept form. Both have a negative slope, BR_1 is steeper than BR_2, and the q_2 intercept is greater for firm 1.

Figure 9.1 illustrates these best-reply functions. Their intersection gives us the Cournot or Nash equilibrium. At this point each firm is maximizing profit, given its belief that its rival is doing the same, a belief that is consistent with actual behavior. That is, this point represents a mutual best reply, and neither firm has an incentive to deviate from this point. The diagram can also be used to visualize the comparative static results that output rises with parameter *a* and falls with parameters b and c.

Another way of deriving best-reply functions and of visualizing the Cournot equilibrium is with isoprofit equations. For firm 1, for example, this describes all combinations of q_1 and q_2 for a constant level of profit ($\pi_1 = k$). In other words, the isoprofit equation includes all q_1-q_2 pairs of points that satisfy

$$\pi_1 = k = aq_1 - bq_1^2 - bq_1q_2 - cq_1. \tag{9.8}$$

We obtain the isoprofit equation by solving for q_2: $q_2 = (aq_1 - cq_1 - bq_1^2 - k)/(bq_1)$, which is much like an indifference curve in consumer theory and an isoquant in production theory. A series of isoprofits for firm 1 are graphed in **Figure 9.2** for different values of k. Notice that they are concave to the q_1 axis and that firm 1's profits rise as we move to lower isoprofits (i.e., $k_B > k_A$). The reason for this is that for a given value of q_1 (e.g., q_1'), firm 1's profits increase as q_2 falls (from q_2' to q_2''), as indicated by equation (9.8). The same can be done for firm 2; the only difference is that its isoprofit will be concave to the q_2 axis.

Isoprofits and the definition of a best-reply can be used to derive a firm's best-reply function. To illustrate, consider firm 1's problem when $q_2 = q_2'$, as described in **Figure 9.3**. To obtain firm 1's best reply, firm 1 will chose the level of output that maximizes its profits, given the constraint that $q_2 = q_2'$. That will put the firm on the lowest possible isoprofit, which occurs at the tangency point A. Notice that this is simply a constrained optimization problem. Similarly, when $q_2 = q_2''$, the tangency point is at B. The locust of these tangencies for all values of q_2 generates firm 1's best-reply function. The same approach can be used to derive firm 2's best-reply function.

Figure 9.4 describes the Cournot equilibrium with respect to best-reply functions and isoprofits. At the equilibrium, it is clear that each firm is maximizing its profit given that its rival is producing at the equilibrium level of output. Thus, this is a Nash equilibrium because it is a mutual best reply and neither firm has an incentive to deviate. Notice, however, that each firm can earn higher profits if they both cut production and move into the shaded, lens-shaped region, an issue that will be discussed later in the book.

9.1.2 The Cournot Model with Two Firms and Asymmetric Costs

We next consider the Cournot model when firm 1 has a cost advantage over firm 2, where $c_1 < c_2$. All other conditions remain the same so that firm profits are

$$\pi_1 = aq_1 - bq_1^2 - bq_1q_2 - c_1q_1, \quad (9.9)$$
$$\pi_2 = aq_2 - bq_2^2 - bq_1q_2 - c_2q_2. \quad (9.10)$$

We obtain the Cournot equilibrium using the same method as before. We solve the first-order conditions of profit maximization and solve them simultaneously for output, and plug these optimal values into the demand and profit functions to obtain the Cournot equilibrium. In this case, the first-order conditions are[3]

$$\frac{\partial \pi_1}{\partial q_1} = a - 2bq_1 - bq_2 - c_1 = 0, \quad (9.11)$$
$$\frac{\partial \pi_2}{\partial q_2} = a - bq_1 - 2bq_2 - c_2 = 0. \quad (9.12)$$

The Cournot equilibrium is

$$q_1^* = \frac{a - 2c_1 + c_2}{3b}, \quad (9.13)$$
$$q_2^* = \frac{a + c_1 - 2c_2}{3b}, \quad (9.14)$$
$$p^* = \frac{(a + c_1 + c_2)}{3}, \quad (9.15)$$
$$\pi_1^* = \frac{(a - 2c_1 + c_2)^2}{9b}, \quad (9.16)$$
$$\pi_2^* = \frac{(a + c_1 - 2c_2)^2}{9b}. \quad (9.17)$$

Although firms face different costs, the model is symmetric because the interchangeability condition holds for the first-order conditions. In other words, we can write firm i's first-order condition as

$$\frac{\partial \pi_i}{\partial q_i} = a - 2bq_i - bq_j - c_i = 0, \quad (9.18)$$

When this occurs, we can write the Nash equilibrium more compactly as

$$q_i^* = \frac{a - 2c_i + c_j}{3b}, \quad (9.19)$$
$$p^* = \frac{\left(a + c_i + c_j\right)}{3}, \quad (9.20)$$
$$\pi_i^* = \frac{(a - 2c_i + c_j)^2}{9b}, \quad (9.21)$$

Note that as c_i approaches c_j (value c), the solution approaches the Cournot equilibrium with symmetric costs found in equations (9.3 – 9.5). The key insight gained by studying the model with asymmetric costs is that firm i's output and profit levels rise as rivals costs increase. Thus, by having lower costs, firm 1 has a strategic advantage over firm 2: $q_1 > q_2$ and $\pi_1 > \pi_2$ in equilibrium.

This can be seen graphically with best-reply functions. Again, they are derived by solving each firm's first-order conditions for q_2.

$$BR_1: q_2 = \frac{a - c_1}{b} - 2q_1, \quad (9.22)$$

[3] Notice that the second-order condition of profit maximization holds, because the second derivative of the profit function for each firm is -2b < 0.

$$BR_2\colon\ q_2 = \frac{a - c_2}{2b} - \frac{1}{2}q_1, \qquad (9.23)$$

Compared to the symmetric case, the slopes are unchanged but the distance between q_2 intercepts for firms 1 and 2 is greater. As described in **Figure 9.5**, this increases the equilibrium value of q_1 and decreases the equilibrium value of q_2.

This model also provides one explanation why firm 1 may have a monopoly position. Notice that as c_2 rises, the equilibrium values of q_1 rises and q_2 falls. For sufficiently high c_2, the best-reply functions intersect at a negative value of q2. This is described in **Figure 9.6**. Because output must be non-negative, firm 2 will shut down ($q_2^* = 0$), leaving firm 1 in a monopoly position. In this case, the Cournot equilibrium is the same as the monopoly solution, with $q_1^* = (a – c_1)/(2b)$, $p^* = (a + c_1)/2$, and $\pi_1^* = (a – c_1)^2/(4b)$.

9.1.3 The Cournot Model with n Firms and Symmetric Costs

We next consider the Cournot model that is symmetric but has n firms. This allows us to see how the equilibrium changes as we go from a monopoly market with n = 1 to a perfectly competitive market where n approaches infinity.

Because this is a symmetric problem, we can describe the profit function for firm i, where i now represents any firm from 1 through n.

$$\pi_i = (p – c)q_i = a – b(q_1 + q_2 + q_3 + \ldots + q_n)q_i – cq_1. \qquad (9.24)$$

For notational convenience, we can rewrite equation (9.20) as

$$\pi_i = (p – c)q_i = a – b(q_i + Q_{-i}) – cq_1, \qquad (9.25)$$

where Q_{-i} is the sum of rival output. With this notation, total industry production $Q = q_i + Q_{-i}$, and $Q = q_i$ in the monopoly case. The first-order conditions for this problem is

$$\frac{\partial \pi_i}{\partial q_i} = a - 2bq_i - bQ_{-i} - c = 0, \qquad (9.26)$$

Given that this is a symmetric problem, output will be the same for each firm and $Q_{-i} = (n – 1)q_i$ in equilibrium.[4] Substituting and solving for q_i in equation (9.25) gives the Cournot equilibrium

$$q_i^* = \frac{a - c}{b(n+1)}, \qquad (9.27)$$

$$p^* = \frac{a}{n+1} + c\ \frac{n}{n+1}, \qquad (9.28)$$

$$\pi_i^* = \frac{(a-c)^2}{b(n+1)^2}, \qquad (9.29)$$

$$Q^* = nq_i^* = \frac{a-c}{b}\frac{n}{n+1}, \qquad (9.30)$$

This model produces two important implications.

- When n = 1, the Cournot equilibrium is the monopoly outcome, where $q_i^* = Q^* = (a – c)/(2b)$, $p^* = (a + c)/2$, and $\pi_i^* = (a – c)^2/(4b)$.
- As n approaches infinity, the Cournot equilibrium approaches the perfectly competitive outcome, where $Q^* = (a – c)/b$, $p^* = c$, and $\pi_i^* = 0$.

This demonstrates an important result in oligopoly theory, the *Cournot Limit Theorem*: the Cournot equilibrium approaches the competitive equilibrium as n approaches infinity.

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[4] This is true only in equilibrium, however. We can set $Q_{-i} = nq_{-i}$ in the first-order condition because optimal values are embedded in it. Because this need not be true out of equilibrium, we cannot make this substitution at an earlier stage in the analysis [i.e., into the profit function in equation (9.22)].

The Cournot model also has intuitive implications regarding the effect of n on allocative efficiency. We illustrate this in **Figure 9.7**, where the monopoly outcome is represented by price p_m and quantity Q_m and the perfectly competitive outcome by p_{pc} and Q_{pc}. As seen in chapter 6, total surplus (consumer plus producer surplus) is maximized in perfect competition and equals area $a p_{pc}$B. For a monopolist, total surplus equals area $a p_{pc}$EA, implying a deadweight loss or an efficiency loss equal to area ABE. In the Cournot model, the price-quantity pair moves from point A to point B as n changes from 1 to infinity, and the efficiency loss goes to zero. In other words, efficiency improves with free entry, and competitive behavior occurs only when there are many firms.

If this were generally true, there would be little left to say about the effect of market structure on allocative efficiency. Unfortunately, this is not the case, which will become apparent as we progress through this chapter and the rest of the book.

9.1.4 The Bertrand Model

The second duopoly model of importance resulted from a review of the Cournot model by Joseph Bertrand (1883). Bertrand questioned the realism of Cournot's assumption that firms compete in output. Bertrand argued that in most markets, firms set prices and let the market determine the quantity sold. In a review of Cournot and Bertrand's work, Fisher (1898, p. 126) supports Bertrand's concern, stating that price competition is more common in business and is, therefore, a more "natural" choice variable. Recall that for a monopolist, the quantity-price outcome is the same whether the firm chooses output or price as the choice variable. We will see that this is not the case in an oligopoly market, however.

To make it easier to compare and contrast the Cournot and Bertrand models, we assume the same demand and cost conditions. We begin by assuming a duopoly setting and symmetric costs. Because firms compete in price in the Bertrand model, however, we are interested in the demand function as this has the choice variable on the right hand side of the demand equation. Solving for output, the demand function is $Q = (a - p)/b$. Note that the price intercept is a, the quantity intercept is a/b, the slope of the inverse demand function is b, and the slope of the demand function is 1/b. In this case, firm i's problem is: max $\pi_i(p_i, p_j)$ with respect to p_i instead of q_i. Once firms set prices, consumers determine quantity demanded.

It turns out that the solution to the problem is also a Nash equilibrium, which is called a Bertrand equilibrium or a Nash equilibrium in prices to a homogeneous goods duopoly game. The formal characteristics of the game are as follows.

- Players: Firms 1 and 2.
- Strategic Variable: Firm i chooses non-negative values of p_i.
- Payoffs: Firm i's payoffs are profits, $\pi_i(p_i, p_j) = pq_i - cq_i = (p_i - c)q_i$.

If the profit function of each firm were differentiable, we would find the Nash equilibrium using the same approach that we used to find the Cournot equilibrium. We would calculate the first-order conditions with respect to price for each firm and solve them simultaneously to obtain Nash equilibrium prices. Unfortunately, the firm's demand and, therefore, profit functions are discontinuous.

To see this discontinuity, consider firm i's demand function. Notice that because the products are homogeneous, consumers will always purchase from the cheapest seller. If prices are the same, $p \equiv p_i = p_j$, consumers are indifferent between purchasing from firms 1 and 2. In this case, the usual assumption is that half of the consumers purchase from firm 1 and the other half from firm 2. Under these conditions, the quantity demanded faced by firm i is

$$q_i = \begin{cases} 0 \quad if\ p_i > a \\ 0 \quad if\ p_i > p_j \\ \frac{a-p}{2b} \quad if\ p_i = p_j < a \\ \frac{a-p_i}{b} \quad if\ p_i < \min(a,\ p_j) \end{cases} \tag{9.31}$$

The discontinuity is easy to see in the graph of firm i's demand, $d_i \equiv q_i$, found in **Figure 9.8** for a given $p_j < a$. It shows that demand is 0 when $p_i > p_j$ and equals the market demand, $(a - p_i)/b$ when $p_i < p_j$. By assumption, demand is half the market demand, $(a - p_i)/(2b)$, when $p_i = p_j$. Thus, the firm's demand is the hatched lines and the $(a - p_j)/(2b)$ when $p_i = p_j$. Because costs are linear, the discontinuity in demand produces a discontinuity in profits, as illustrated in **Figure 9.9**. Recall that for a monopolist, profits are quadratic for linear demand and cost functions. In this duopoly case, however, firm i's profits are quadratic when $p_i < p_j$, equaling $(a - p_i)(p_i - c)/b$, but are 0 when $p_i > p_j$. When $p_i = p_j$, profits are split evenly between firms, equaling $(a - p)(p - c)/(2b)$.

Given that profit functions are discontinuous when firms charge the same price, we need to use the characteristics of a Nash equilibrium to identify the Bertrand solution. Recall that players will have no incentive to deviate at the Nash equilibrium. This occurs only when $p_i = p_j = c$ for $c < a$. Because the proof provides a deeper insight into a Nash equilibrium, we provide it below.

Proof: Consider all relevant possibilities where p_i and p_j are positive but less than $a > c$.

- $p_i > p_j > c$: This is not a Nash equilibrium because firm i can increase its profits by setting its price between p_j and c (assuming the medium of exchange is infinitely divisible).
- $p_i > p_j < c$: This is not a Nash equilibrium because $\pi_j < 0$ and firm j can increase its profits by shutting down.
- $p_i = p_j > c$: This is not a Nash equilibrium because each firm can increase profits by cutting its price below its rival's price and above c.
- $p_i = p_j < c$: This is not a Nash equilibrium because $\pi_i < 0$ and $\pi_j < 0$. Both firms can increase profits by shutting down.
- $p_i = p_j = c$: This is a Nash equilibrium because neither firm can increase its profits by changing its price or by shutting down. Q.E.D.

The only outcome where neither firm has an incentive to deviate is when $p_i = p_j = c$, the Nash or Bertrand equilibrium to this game. The intuition behind this result is that firms will keep *undercutting* the price of its rival until price equals marginal cost. Notice that this produces a perfectly competitive outcome: $p = c$, $\pi_i = 0$, and $Q = (a - c)/b$. Comparative static results are the same as in perfect competition. That is, the long run price changes with marginal cost, and industry production increases with demand and falls with marginal cost.

Recall that in the monopoly case that the solution was the same, whether the firm maximized profit over output or price. This is not generally the case with oligopoly. Although the Bertrand model is identical in every way to the Cournot model except that price is the choice variable instead of ouput, the outcome is dramatically different. This illustrates that a firm's strategic choice, as well as demand and cost conditions, affect the Nash equilibrium in an oligopoly setting.

We next extend the Bertrand model to the case where $n > 2$. It is easy to verify that the Bertrand model with symmetric costs produces the perfectly competitive result as long as $n > 1$. That is, price undercutting will lead to price competition that is so fierce that only 2 or more firms are needed to generate a perfectly competitive outcome that is allocatively efficient. Notice that this result is dramatically different from the Cournot outcome, where many competitors are required for the market to be allocatively efficient. Because this result is extreme and generally inconsistent with reality, it is called the *Bertrand paradox*.

One way for a firm to avoid the Bertrand paradox and earn economic profit in a Bertrand setting is to have a competitive cost advantage over its rivals. Returning to the duopoly case, let $c_1 < c_2$. In this case, undercutting will produce an outcome where firm 1 undercuts p_2 by charging the highest possible price that is lower than c_2, assuming that c_2 is less than firm 1's simple monopoly price (p_m). When c_2 is greater than p_m, firm 1 will set its price to p_m. Thus, there will be only one seller in the market, but its price may be below its simple monopoly price. Note that this is different from the Cournot model, where both the high and low cost firms coexist. An important implication of the Bertrand model is that it shows how the presence of a potential, although higher cost, entrant can reduce the price charged by a monopolist. Later in the chapter, we show how product differentiation can also be used to lower price competition in a Bertrand setting.

9.2 Cournot and Bertrand Models with Differentiated Products

We begin with a simple model of product differentiation, one where products differ on a number of characteristics and consumers have a taste for variety and consume a variety of brands. For example, most consumers prefer to frequent a variety of restaurants, rather than eating at the same Chinese, Italian, or Mexican restaurant over and over again. Later in the chapter we consider models with horizontal and vertical product differentiation where consumers have a preference for one brand over another. Our goals are to understand how product differentiation affects equilibrium prices, production, profits, and **allocative efficiency**. In this chapter, we assume that firms have already chosen product characteristics. Thus, the degree of product differentiation is predetermined. Later in the book, we will analyze models where product characteristics are choice variables.

To keep things simple, we assume a duopoly market. Firms face the same variable costs, although fixed costs may vary by firm. Thus, the cost difference between brands is due to a difference in set-up costs, not marginal costs.

9.2.1 The Cournot Model

Consider a Cournot duopoly where firms produce different products that meet consumer demand for variety. This can be modeled with the following set of inverse demand functions, which is similar to that of Dixit (1979) and Singh and Vives (1984):[5]

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[5] Dixit (1979) shows that this demand system derives from the utility function $U(q_1, q_2) = \alpha_1 q_1 + \alpha_2 q_2 - (\beta_1 q_1^2 + \beta_2 q_2^2 + 2\gamma q_1 q_2)/2$, where $a = \alpha_1 = \alpha_2$, $\beta_1 = \beta_2 = 1$, and $d = \gamma$.

$$p_1 = a - q_1 - dq_2, \quad (9.32)$$
$$p_2 = a - q_2 - dq_1. \quad (9.33)$$

Recall that in the homogeneous goods case that there was a single parameter, b, in front of the output of each firm. In the current model, the parameter in front of the firm's own output is 1 and the parameter in front of its rival output is d. Notice that when d = 1, products are homogeneous; when d = 0, each firm is a monopolist. Thus, d can be thought of as an index of product differentiation, because a decrease in d implies a greater degree of product differentiation. Firm i's total cost equation is $cq_i – F_i$, where F_i is the firm's fixed cost. Thus, firm i's profits are $\pi_i(q_i, q_j) = (p_i – c)q_i = (a – q_i – dq_j)q_i – cq_i – F_i$.

The profit function is differentiable, so the Cournot equilibrium can be derived in the same way as in the homogeneous goods case. That is, we derive the first-order conditions and solve them simultaneously for output. The first-order conditions are[6]

$$\frac{\partial \pi_1}{\partial q_1} = a - 2q_1 - dq_2^* - c = 0, \quad (9.34)$$
$$\frac{\partial \pi_2}{\partial q_2} = a - dq_1^* - 2q_2 - c = 0. \quad (9.35)$$

Notice that the interchangeability condition holds. Thus, the Cournot equilibrium is symmetric:

$$q_i^* = \frac{a - c}{2 + d}. \quad (9.36)$$
$$p_i^* = \frac{(a + c + cd)}{2 + d}, \quad (9.37)$$
$$\pi_i^* = \frac{(a - c)^2}{(2 + d)^2} - F_i. \quad (9.38)$$

We graph the best-reply functions and the Cournot equilibrium in **Figure 9.10**, which we will later use to compare with the Bertrand equilibrium when there is product differentiation.

The main reason for studying this model is to determine how product differentiation affects the Cournot equilibrium.[7] The important implications of the model with product differentiation are:

- The equilibrium converges to the homogeneous Cournot equilibrium as d approaches 1 (i.e., Figures 9.1 and 9.10 are the same).
- The equilibrium converges to the monopoly equilibrium as d approaches 0.
- Greater product differentiation (i.e., lower d) leads to higher prices and a reduction in allocative efficiency.

The price effect is illustrated in **Figure 9.11**, which shows that the Cournot price increases with greater product differentiation (i.e., as d decreases from 1 to 0). The fact that product differentiation can reduce price competition is called the *principle of product differentiation*.

9.2.2 The Bertrand Model

We now want to analyze the Bertrand model when firms face the same demand structure as in the differentiated Cournot model that are found in equations (9.29) and (9.30). Instead of the inverse demand functions, however, we are interested in the demand functions so that the choice

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[6] Notice that the second-order condition of profit maximization holds, because the second derivative of the profit function for each firm is -2 < 0.

[7] The effects of a change in marginal cost and a change in the demand intercept are the same as in the case with homogeneous goods.

variables are on the right hand side of the equation. Solving the inverse demand functions in (9.28) and (9.29) yield the following demand functions:

$$q_1 = \alpha - \beta p_1 + \delta p_2, \tag{9.39}$$

$$q_2 = \alpha - \beta p_2 + \delta p_1, \tag{9.40}$$

where $\alpha \equiv a(1 - d)/x$, $\beta \equiv 1/x$, $\delta \equiv d/x$, and $x \equiv (1 - d^2)$.[8] Note that because d ε(0, 1) for there to be product differentiation, $\beta > \delta$. With this demand system, firm i's profits are $\pi_i(p_i, p_j) = (p_i - c)q_i - F_i = (p_i - c)(\alpha - \beta p_i + \delta p_j) - F_i$.

Notice that with product differentiation, firm i's profit function is differentiable in p_i. This enables us to derive the Nash equilibrium using the same method as in the Cournot model. In this case, we derive each firm's first-order condition with respect to its own price for a given rival price and solve the system of equations. This generates Nash or Bertrand equilibrium prices. The first-order conditions are[9]

$$\frac{\partial \pi_1}{\partial p_1} = \alpha - 2\beta p_1 + \delta p_2 + \beta c = 0, \tag{9.41}$$

$$\frac{\partial \pi_2}{\partial p_2} = \alpha + \delta p_1 - 2\beta p_2 + \beta c = 0. \tag{9.42}$$

Solving these equations simultaneously for p_1 and p_2 yields the Bertrand equilibrium prices. Substituting them into the demand and profit functions gives their equilibrium values. Given that the interchangeability condition is met, the model is symmetric and the Bertrand equilibrium is

$$p_i^* = \frac{\alpha + \beta c}{2\beta - \delta}. \tag{9.43}$$

$$q_i^* = \frac{\beta(\alpha - c(\beta - \delta))}{2\beta - \delta}, \tag{9.44}$$

$$\pi_i^* = \frac{\beta[\alpha - c(\beta - \delta)]^2}{(2\beta - \delta)^2} - F_i. \tag{9.45}$$

Because this model is different from the previous Bertrand models, we also graph the best-reply and isoprofit functions. Solving each firm's first-order condition for p_2, gives the best replies

$$BR_1: p_2 = -\frac{\alpha + \beta c}{\delta} + \frac{2\beta}{\delta} p_1, \tag{9.46}$$

$$BR_2: p_2 = \frac{\alpha + \beta c}{2\beta} + \frac{\delta}{2\beta} p_1, \tag{9.47}$$

The best-reply functions are linear, but unlike the Cournot model, they have a positive slope. For the model to be stable, an issue that we discuss in the appendix, BR_1 will be steeper than BR_2 (i.e. $\beta > \delta/2$), The best-reply functions and the isoprofits are illustrated in **Figure 9.12**, along with the Bertrand equilibrium prices that occur where the best-reply functions intersect. In this model, notice that both firms are better off if they raise prices and move into the shaded, lens shaped region. Thus, both firms earn higher profits when prices are higher.

In a previous section we saw that one way for a firm to avoid the Bertrand paradox is to gain a cost advantage over its competitors. Another way is for firms to differentiate their

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[8] Detailed derivations can be found in Shy (1995, 162-163).

[9] Notice that the second-order conditions of profit maximization hold, because the second derivative of the profit function for each firm is $-2\beta < 0$.

products. This is easy to see when we set fixed costs to zero and write the profit function in terms of the original parameters *a* and d from the Cournot model.

$$\pi_i^* = \frac{(a-c)^2(1-d)}{(2-d)^2(1+d)}. \tag{9.48}$$

Recall that *a* is the price intercept of demand and d indexes product differentiation – products are homogeneous when d = 1 and each firm is a monopolist when d = 0. Thus, profits are zero when products are perfectly homogeneous, the simple Bertrand outcome. In addition, profits approach monopoly profits as d approaches 0. This verifies the principle of product differentiation.

9.2.3 The Bertrand Model and Horizontal and Vertical Differentiation

In the previous section, we analyzed the case where consumers who have a taste for variety consume several different brands within a given market. In other markets, however, consumers have a strong preference for one brand over another. For example, when a red and a blue Honda Civic are priced the same and are homogeneous in every other way, some consumers will prefer the red Civic and others the blue Civic. Thus, these different colored cars are horizontally differentiated. Alternatively, everyone will prefer a Craftsman brand wrench to a Champion brand wrench when priced the same. Both brands are sold by Sears, but Craftsman tools are made from harder steel and have a longer (lifetime) guarantee. Thus, Craftsman tools are of undeniably higher quality. Products such as these are vertically differentiated.

As discussed in the previous chapter, the classic models that account for this type of differentiation are the models by Hotelling (1929) and by Mussa and Rosen (1978). Hotelling provides a simple linear city model of horizontal product differentiation, while Mussa and Rosen model vertical differentiation. The main advantage of these models is that they produce linear demand functions, making it easier to continue our analysis of the relationship between product differentiation and the Bertrand equilibrium. They also provide a theoretical framework for later analysis when firms compete in other dimensions, such as product characteristics and advertising.

9.2.3.1 The Bertrand Model with Horizontal Differentiation (θ)

Recall from the Hotelling model discussed in Chapter 8 that brands differ in terms of a single characteristic (θ) and that consumer have different tastes, with some consumers preferring brands with high levels of θ and others prefer brands with low levels of θ. This model is represented by a linear city that is assumed to be of unit length, with the location on main street (θ) ranging from location 0 to 1. Consumers are uniformly distributed along main street. Two supermarkets (1 and 2) compete for consumer business, with store 1 located at θ_1 and firm 2 locating at θ_2, with $0 \leq \theta_1 \leq ½ \leq \theta_2 \leq 1$.

Supermarkets 1 and 2 are viewed as being homogeneous when they both locate at ½ but are viewed as being increasingly differentiated as they move further and further apart. With positive transportation costs (t), consumers will prefer the store closest to home. To simplify the analysis, we assumes that markets are covered (i.e., no consumer refrains from purchase) and that consumers have unit demands (i.e., each consumer buys just one unit of brand 1 or brand 2 per time period).

As we saw in Chapter 8, these assumptions produce the following linear demand functions.

$$q_1 = \frac{N[t(\theta_2 - \theta_1) - p_1 + p_2]}{2t}, \tag{9.49}$$

$$q_2 = \frac{N[t(\theta_2 - \theta_1) + p_1 - p_2]}{2t}, \tag{9.50}$$

where N is the number of consumers. Note that demand for firm i's brand increases with an increase in the number of consumers, a decrease in transportation costs, a decrease in the firm's own price, and an increase in its rival's price. The model also shows that demand increases as firms move further apart. For now, however, we assume that store location is fixed or predetermined. In this model, firm i's profits are $\pi_i(p_i, p_j) = (p_i - c)q_i - F_i = (p_i - c)\{N[t(\theta_2 - \theta_1) - p_i + \delta p_j]\}/(2t) - F_i$.

As in the previous case, firm i's profit function is differentiable in p_i, enabling us to derive in the Nash equilibrium by differentiation. The first-order conditions are[10]

$$\frac{\partial \pi_1}{\partial p_1} = \frac{N[t(\theta_2 - \theta_1) - 2p_1 + p_2 + c]}{2t} = 0, \tag{9.51}$$

$$\frac{\partial \pi_2}{\partial p_2} = \frac{N[t(\theta_2 - \theta_1) - 2p_2 + p_1 + c]}{2t} = 0. \tag{9.52}$$

Notice that the problem is symmetric, because the interchangeability condition holds. Solving the first-order conditions for prices and substituting them into the demand and profit functions yields the Bertrand equilibrium.

$$p_i^* = c + t(\theta_2 - \theta_1), \tag{9.53}$$

$$q_i^* = N(\theta_2 - \theta_1)/2, \tag{9.54}$$

$$\pi_i^* = \frac{Nt(\theta_2 - \theta_1)^2}{2} - F_i. \tag{9.55}$$

This also verifies the principle of product differentiation, because competition falls and profits rise as firms 1 and 2 move further apart (i.e, as $\theta_2 - \theta_1$ increases), an issue we take up in a later chapter. As in the previous model of product differentiation, the model produces positively sloped best-reply functions. Deriving and graphing the best-reply functions is left as an excise at the end of the chapter.

9.2.3.2 The Bertrand Model with Vertical Differentiation (ϕ)

Next, we consider a duopoly model with vertical differentiation where firm 1 produces a brand of higher quality or reliability. Recall from Chapter 8 that z_i indexes the quality of brand i, where $z_1 > z_2 > 0$, and the degree of vertical differentiation is $z \equiv z_1 - z_2$. Consumers all prefer the high quality brand but some have a stronger preference for quality than others. A consumer's preference for quality is represented by ϕ, and the diversity of consumer tastes ranges from ϕ_L to ϕ_H, with $\phi_H > \phi_L > 0$ and $\phi_H - \phi_L = 1$. Later we will see that another constraint will be important, that $\phi_H > 2\phi_L > 0$.

From the previous chapter, we know that the Mussa and Rosen model produces linear demand functions that take the following form.

$$q_1 = \frac{N(z\phi_H - p_1 + p_2)}{z}, \tag{9.56}$$

$$q_2 = \frac{N(-z\phi_L + p_1 - p_2)}{z}. \tag{9.57}$$

Demand for firm i's brand increases with an increase in the number of consumers, an increase in the degree of vertical differentiation, a decrease in the firm's own price, and an increase in its

[10] Notice that the second-order conditions of profit maximization hold, because the second derivative of the profit function for each firm is -N/t < 0.

rival's price. Demand also increases with the degree of product differentiation. For now, however, product quality is fixed. Assuming symmetric variable costs, firm i's profits are $\pi_i(p_i, p_j) = (p_i - c)q_i - F_i$. Profit functions are differentiable, and firm first-order conditions are[11]

$$\frac{\partial \pi_1}{\partial p_1} = \frac{N(z\phi_H - 2p_1 + p_2 + c)}{z} = 0, \quad (9.58)$$

$$\frac{\partial \pi_2}{\partial p_2} = \frac{N(-z\phi_L - 2p_2 + p_1 + c)}{z} = 0. \quad (9.59)$$

Because $\phi_H \neq -\phi_L$, the problem is not symmetric. Solving the first-order conditions for prices and substituting them into the demand and profit functions yields the Bertrand equilibrium when differentiation is vertical.

$$p_1^* = c + \frac{z(2\phi_H - \phi_L)}{3} > p_2^* = c + \frac{z(\phi_H - 2\phi_L)}{3}. \quad (9.60)$$

$$q_1^* = \frac{N(2\phi_H - \phi_L)}{3} > q_2^* = \frac{N(\phi_H - 2\phi_L)}{3}. \quad (9.61)$$

$$\pi_1^* = \frac{Nz\left(2\phi_H - \phi_L\right)^2}{9} - F_1; \qquad \pi_2^* = \frac{Nz\left(\phi_H - 2\phi_L\right)^2}{9} - F_2 \quad (9.62)$$

Notice that for firm 2 to produce a positive level of output, $\phi_H > 2\phi_L$. Thus, this constraint must be imposed for there to be two firms in the market.

This model produces several interesting results. First, the high quality firm sells more output and at a higher price. The high quality firm will also have a strategic advantage as long as the difference in fixed costs is not too great. Finally, the principle of differentiation is verified, as prices and profits increase as the degree of production differentiation increases (i.e., as z increases).

Because this is our first asymmetric model, we derive and graph the best-reply functions. Recall that we can obtain the best-reply functions by solving each firm's first-order condition with respect to p_2.

$$BR_1: p_2 = -(c + z\phi_H) + 2p_1, \quad (9.63)$$

$$BR_2: p_2 = \frac{c - z\phi_L}{2} + \frac{1}{2}p_1. \quad (9.64)$$

The best-reply functions are linear and are illustrated in **Figure 9.13**. This verifies that the high quality producer will charge a higher price than the low quality producer.

9.3 The Cournot-Bertrand Model

An important concern with the Cournot and Bertrand models is that they take the strategic variable as given. That is, firms either compete in output (Cournot) or in price (Bertrand). This raises the question – why can't one firm compete in output and the other firm compete in price? After all, Kreps and Scheinkman (1983) argue that it is "whitless" to criticize the choice of strategic variable, as it is an empirical question whether or not firms compete in output or in price.

[11] The second-order conditions of profit maximization hold, because the second derivative of the profit function for each firm is $-2/z < 0$.

Although firms compete in either output or in price in most industries, a mixture of Cournot and Bertrand behavior is another possibility and is observed in the market for small cars. In each period, Honda and Subaru dealers establish inventories and adjust price to reach their sales goals, while Saturn and Scion dealers fill consumer orders at a fixed price. Given that such behavior is observed, Tremblay et al. (2008) develop a Cournot-Bertrand model with product differentiation, where consumers have a demand for variety, firm 1 competes in output, and firm 2 competes in price. In their model, the demand system must have the strategic variables (q_1 and p_2) on the right hand side of the demand equation.

$$p_1 = a - q_1 + bp_2, \tag{9.65}$$
$$q_2 = a - p_2 + dq_1, \tag{9.66}$$

where b and d are positive demand parameters, and d < 2.[12] As before, firm i's total cost equation is $cq_i - F_i$, and its profits are $\pi_i(q_i, q_j) = (p_i - c)q_i - F_i$.

One thing to note is that the problem is naturally asymmetric because firms have different choice variables. This is clear from the firms' profit maximization problems:

$$\max_{q_1} \ \pi_1 = p_1q_1 - cq_1 - F_1 = (a - q_1 + bp_2)q_1 - cq_1 - F_1, \tag{9.67}$$
$$\max_{p_2} \ \pi_2 = p_2q_2 - cq_2 - F_2$$
$$= p_2(a - p_2 + dq_1) - c(a - p_2 + dq_1) - F_2. \tag{9.68}$$

That is, firm 1 maximizes profit with respect to output, and firm 2 maximizes profit with respect to price. One can see from the first-order conditions that the interchangeability condition does not hold.[13]

$$\frac{\partial \pi_1}{\partial q_1} = a - 2q_1 + bp_2 - c = 0, \tag{9.69}$$
$$\frac{\partial \pi_2}{\partial p_2} = a - 2p_2 - dq_1 + c = 0. \tag{9.70}$$

Because the solution is rather complex, we simplify the model by normalizing marginal cost to zero. With this assumption, p_i can be thought of as the difference between the price and marginal cost. The Nash equilibrium to this game is

$$p_1^* = \frac{a(2+d)}{4+bd} > p_2^* = \frac{a(2-d)}{4+bd}, \tag{9.71}$$
$$q_1^* = \frac{a(2+d)}{4+bd} > q_2^* = \frac{a(2-d)}{4+bd}, \tag{9.71}$$
$$\pi_1^* = \frac{a^2(2+d)^2}{(4+bd)^2} - F_1; \quad \pi_2^* = \frac{a^2(2-d)^2}{(4+bd)^2} - F_2, \tag{9.72}$$

This outcome has two interesting features. First, firm 1 charges a higher price and produces more output. Second, firm 1 has a strategic advantage over firm 2 as long the difference in fixed costs is not too great. For this reason, d must be less than 2 for firm 2 to participate.

To further analyze this model, we describe the Nash equilibrium in terms of best-reply and isoprofit diagrams. We obtain the best reply-functions by solving the first-order conditions for p_2.

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[12] Tremblay et al. (2008) show how this demand system can be derived from a demand system similar to that of equations (9.31) and (9.32).

[13] Notice that the second-order condition of profit maximization holds, because the second derivative of the profit function for each firm is -2 < 0.

$$BR_1: p_2 = \frac{c-a}{b} + \frac{2}{b}q_1, \tag{9.73}$$

$$BR_2: p_2 = \frac{a+c}{2} - \frac{d}{2}q_1, \tag{9.74}$$

The best-reply functions and isoprofits are illustrated in **Figure 9.14**. The natural asymmetry of the model is evident from the fact that firm 1's best reply has a positive slope and firm 2's best reply has a negative slope. Notice that firm 1's isoprofit is convex to the q_1 axis. This is because firm 1's profits increase as p_2 increases. In contrast, firm 2's isoprofit is concave to the p_2 axis because firm 2's profits increase as q_1 falls. These features of best reply functions and isoprofits occur because the model mixes Cournot and Bertrand strategic choices. Finally, notice that both firms are better off if firm 1 reduces production and firm 2 raises price, actions that lessen competition.

9.4 The Choice of Output Versus Price Competition is Endogenous

We have examined the classic Cournot and Bertrand models in homogeneous and differentiated goods markets. We have also developed a Cournot-Bertrand model in a differentiated goods market. These models produce several important conclusions.

- In a market with homogeneous goods, prices and profits are substantially higher in the Cournot model than in the Bertrand model. Although the equilibrium in a monopoly setting is invariant to choosing output or price as the choice variable, the choice of strategic variable makes a big difference when there is more than one firm.
- The perfectly competitive solution is reached in the Bertrand model as long as there are two or more competitors. In contrast, the Cournot solution approaches the perfectly competitive equilibrium only as n approaches infinity.
- Competition diminishes with product differentiation in both the Cournot and Bertrand models, and the two models are much more alike in differentiated goods markets.
- A duopoly model becomes naturally asymmetric when firms compete in different choice variables. In the Cournot-Bertrand model, the firm that chooses to compete in output has a strategic advantage over the firm that chooses to compete in price, as long as the difference in fixed costs is not too great.

These results raise the following question – if firms are given the choice, when is it optimal to compete in output and when is it optimal to compete in price? Clearly, when products are homogeneous and costs are symmetric, output competition is superior to price competition. This issue is more complex when products are differentiated and costs are asymmetric, however, issues that are analyzed by Singh and Vives (1984) and Tremblay et al. (2008). They consider a duopoly model where firms differentiate their products to meet consumer demand for variety. The important feature of their models is that the decision to compete in output or price is endogenous. This leads to four possible outcomes.

1. Cournot (C): Both firms choose to compete in output.
2. Bertrand (B): Bother firms choose to compete in price.
3. Cournot-Bertrand (CB): Firm 1 chooses to compete in output and firm 2 chooses to compete in price.

4. Bertrand-Cournot (BC): Firm 1 chooses to compete in price and firm 2 chooses to compete in output.

We summarize their findings by using the demand system found in equations (9.64) and (9.65). This represents the demand equations for the Cournot-Bertrand model. To investigate the other three cases, the demand system must be rearranged so that firm choice variables are on the right hand side of each firm's demand equation.[14] In order to compare profits for these four cases, we set c = 0, a = 19, b = ½, and d = 3/2.

Table 1 illustrates the results. Regarding fixed costs, superscript C identifies the case where the firm competes in output, and superscript B identifies the case where the firm competes in price. This demand and cost structure clearly produces an asymmetric outcome, but further analysis requires that we make assumptions about fixed costs. If fixed costs are the same regardless of the strategic choice (i.e., $F_1^C = F_1^B$ and $F_2^C = F_2^B$), then both firms are better off competing in output than in price.[15] That is, if firms had the choice, they would always prefer to compete in output because this is the dominant strategy. If this conclusion were universally true, the question is why is price competition so common in the real world?

There are three ways to overturn this conclusion. First, the fixed cost associated with output competition may be substantially higher than the fixed cost associated with price competition (i.e, $F_i^C > F_i^B$). With output competition, a firm must bring a substantial quantity of output to market. Because sales take time, the firm must have a storage facility to hold inventory. A firm that competes in price, however, may fill customer orders only after an order is placed. In the example in Table 9.1, if $F_1^C = 100$ and $F_1^B = 0$, then both firms would prefer price over output competition. Second, Häckner (2000) shows brands are differentiated vertically, price competition will be more profitable than output competition when the quality difference between brands is sufficiently large. Finally, in a dynamic setting price competition is more likely, at least for the follower. We take up this issue in the next chapter.

9.5 Strategic Substitutes and Strategic Complements

An interesting feature of these simple parametric models is that the best-reply functions exhibit a consistent pattern. When firms compete in output, the best reply functions have a negative slope, and when they compete in price with product differentiation, they have a positive slope. Bulow et al. (1985) discovered these patterns and gave them the following names:

1. The strategies of two players are said to be *strategic substitutes* when the best-reply functions have a negative slope.
2. The strategies of two players are said to be *strategic complements* when the best-reply functions have a positive slope.

So far, we have investigated only best-reply functions for price and output, but these definitions apply to all strategic variables.

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[14] Tremblay et al. (2008) show that this demand system produces the following demand structure in the Cournot model: $p_1 = a_1 - d_1q_1 - bq_2$ and $p_2 = a - q_2 - dq_1$. In the Bertrand model: $q_1 = a - p_1 + bp_2$ and $q_2 = a_2 - b_2p_2 + dp_1$. In the Bertrand-Cournot model: $q_1 = a_3 - d_3p_1 - b_3q_2$ and $p_2 = a_4 - d_3q_2 + d_4p_1$. This assumes that $a_1 = (a + ab)$, $a_2 = (a - ad)$, $a_3 = a_1/d_1$, $a_4 = a_2/d_1$, $b_3 = b/d_1$, $d_1 = (1 + bd)$, $d_3 = 1/d_1$, and $d_4 = d/d_1$.

[15] Of course, setup costs may be higher for output competition than for price competition, for example.

In general, whether a strategic variable between two firms is a strategic substitute or complement hinges on how a change in firm j's strategic variable (s_j) affects the marginal returns of firm i's strategic variable (s_i). More formally, given firm i's profit equation, $\pi_i(s_i, s_j)$, which is assumed to be strictly concave in s_i and twice continuously differentiable, marginal returns are defined as $\partial\pi_i/\partial s_i$. The effect of s_j on firm i's marginal returns is $\partial^2\pi_i/\partial s_i\partial s_j \equiv \pi_{ij}$. It turns out that s_i and s_j are strategic complements when $\pi_{ij} > 0$ and are strategic substitutes when $\pi_{ij} < 0$.

The proof follows from the first- and second-order conditions of profit maximization and the application of the implicit-function theorem. Recall that firm i's best-reply function is derived by solving the firm's first order condition for s_i, s_i^{BR}, which is the optimal value of s_i given s_j. Even though we are using a general function, embedded in the first-order condition is s_i^{BR}. Thus, we can use the implicit-function theorem to obtain the slope of firm i's best-reply function, which is

$$\frac{\partial s_i^{BR}}{\partial s_j} = \frac{-\pi_{ij}}{\pi_{ii}}, \qquad (9.75)$$

where $\pi_{ii} \equiv \partial^2\pi_i/\partial s_i^2$, which is negative from the second-order condition of profit maximization. Thus, the sign of $\partial s_i^{BR}/\partial s_j$ equals the sign of π_{ij}. To summarize:

- When $\pi_{ij} < 0$, the best-reply functions have a negative slope and s_i and s_j are strategic substitutes, as in the Cournot model.
- When $\pi_{ij} > 0$, the best-reply functions have a positive slope and s_i and s_j are strategic complements, as in the differentiated Bertrand model.

In the mixed Cournot and Bertrand model developed in section 9.3, notice that $\pi_{12} = b > 0$ and $\pi_{21} = -d < 0$. This verifies that firm 1's best-reply function has a positive slope, and firm 2's best-reply function has a negative slope (Figure 9.14).

9.6 Summary

1. An oligopoly is characterized by a market with a few firms that compete in a strategic setting because a firm's profits and best course of action depend on the actions of its competitors. A duopoly is a special case of oligopoly where there are two firms in the market.

2. In the Cournot model, each firm chooses a level of output that maximizes its profits, given the output of its competitors. The Cournot equilibrium is a Nash equilibrium where each firm correctly assumes that its competitors behave optimally. As the number of firms in a market changes from 1 to many, the Cournot equilibrium changes from monopoly to the perfectly competitive equilibrium.

3. In the Bertrand model with homogeneous goods, each firm chooses its price to maximize profits, given the price of its competitors. The Bertrand equilibrium is a Nash equilibrium where the price equals the perfectly competitive price as long as there are two or more firms in the market. This occurs because each firm will undercut the price of its rivals until the competitive price is reached. This outcome is called the Bertrand Paradox.

4. The interchangeability condition means that the first-order conditions of every firm in a model are interchangeable. When this condition holds, the model is symmetric.

5. According to the *principle of product differentiation*, price competition diminishes as product differentiation increases.

6. In the Cournot-Bertrand model, firm 1 competes in output and firm 2 competes in price. This model produces a naturally asymmetric outcome, and firm 1 has a strategic advantage as long as the difference in fixed costs between firms is not too great.

7. When the choice of strategic variable is endogenous, firms will choose to compete in output as long as there are not substantial cost savings associated with price competition and as long as the degree of vertical product differentiation is not too great.

8. When best-reply functions have a positive slope, as in the Cournot model, the strategic variables between firms are strategic substitutes. When they have a positive slope, as in the Bertrand model, they are strategic complements.

9.7 Review Questions

1. Consider a market with two firms (1 and 2) that face a linear inverse demand function p = a – bQ and a total cost function TC = cq_i, c > 0. Find the Cournot equilibrium output for each firm. How will your answer change if TC = c^2q_i?

2. Consider a market with two firms (1 and 2) that face a linear demand function Q = 24 – p and a total cost function $TC_i = cq_i$, c > 0. Find the Bertrand equilibrium price. How will your answer change if $c_1 = 10$ and $c_2 = 12$?

3. Explain how an increase in the number of firms affects the equilibrium price and allocative efficiency in the homogeneous goods Cournot and Bertrand models.

4. Explain how the principle of product differentiation provides a solution to the Bertrand paradox.

5. In the Bertrand model with horizontal differentiation, explain how the equilibrium changes as t approaches 0. What does this say about the relationship between t and product differentiation?

6. Wal-Mart stores typically locate on the edge of a city, even though potential demand may be greatest at the city's center. Is this a good location strategy?

7. In many markets, high quality brands coexist with low quality brands. If all consumers prefer high to low quality goods, ceteris paribus, why do some firms choose to supply low quality goods to the markets?

8. Consider a market with two firms (1 and 2) where firm 1 competes in output and firm 2 competes in price. Firm 1's inverse demand is $p_1 = 12 – q_1 + p_2$, firm 2's demand is $q_2 = 12 – p_2 + q_1$, $TC_i = cq_i$, $c > 0$. Find Cournot-Bertrand equilibrium price, output, and profit levels for each firm. How does a change in c affect the equilibrium price, output, and profit levels?

9. Assume a duopoly market where firms can choose to compete in output or in price. Provide a simple example where it is optimal for both firms to compete in price instead of output.

10. Show that the strategic variables are strategic substitutes in the Cournot model in section 9.2.1 and are strategic complements in the Bertrand model in section 9.2.2 by evaluating the signs of π_{ij} for each firm and in each model.

11. In the Bertrand model in section 9.2.2, discuss what happens to Nash prices when $\beta = ½$ and $\delta = 1$. Will the model be stable, as described in the appendix, if $\beta = ½$ and $\delta = 2$?

Appendix: Stability of the Cournot and Bertrand Models

The Nash equilibrium in the Cournot and differentiated Bertrand models are stable if in disequilibrium, market forces push the strategic variables toward the equilibrium. Although these models are static, the idea is that if we start at a disequilibrium point, then the adjustment process will converge to the Nash equilibrium.

First, we consider the Cournot model developed in section 9.2.1. The model is stable when BR_1 is steeper than BR_2, as in Figure 9.15. To see this, assume that firm 1 chooses a disequilibrium point, q_1'. Firm 2's best reply to q_1' is q_2''. When firm 2 chooses q_2'', firm 1's best response is q_1'''. Thus, the adjustment process moves from point A, to B, to C in the graph, a process that continues until the Nash equilibrium is reached. At equilibrium, Firm 1's best reply to q_2^* is q_1^*, and firm 2's best reply to q_1^* is q_2^*. The model is unstable when BR_1 is flatter than BR_2, however, as illustrated in Figure 9.16. In this case, when starting at q_1' the adjustment process moves away from the Nash equilibrium.

We now investigate this more generally. Recall that the slopes of the best-reply functions are $\partial q_1^{BR}/\partial q_2 = – \pi_{12}/\pi_{11}$ for firm 1 and $\partial q_2^{BR}/\partial q_1 = – \pi_{21}/\pi_{22}$ for firm 2. In the graph with q_2 on the vertical axis, the slope of firm 1's best reply is $– \pi_{11}/\pi_{12}$. Thus, stability in the Cournot model requires that $|– \pi_{11}/\pi_{12}| > |– \pi_{21}/\pi_{22}|$. Because $\pi_{ii} < 0$ and $\pi_{ij} < 0$, we can rewrite the stability condition as $\pi_{11}\pi_{22} – \pi_{12}\pi_{21} > 0$. In the example from section 9.2.1, $\pi_{11} = \pi_{22} = – 2$ and $\pi_{12} = \pi_{21} = – d$. Thus, the slope of firm 1's best reply is $– 2/d$, the slope of firm 2's best reply is $– d/2$, and the stability conditions is $\pi_{11}\pi_{22} – \pi_{12}\pi_{21} = 2 – d > 0$. Thus, the model is stable when $d < 2$.

Next, we consider the differentiated Bertrand model developed in section 9.2.2. The model is stable when BR_1 is steeper than BR_2, as in Figure 9.17. If we begin at a disequilibrium point, $p_1{}'$, firm 2's best reply is $p_2{}''$. When firm 2 chooses $p_2{}''$, firm 1's best response is $p_1{}'''$, etc. Thus, the adjustment process moves from point A, to B, to C and converges to the Nash equilibrium. This model is unstable, however, when BR_1 is flatter than BR_2, as in Figure 9.17. In this case, the adjustment process moves away from the Nash equilibrium.

In the example from section 9.2.2, $\pi_{11} = \pi_{22} = -2\beta$ and $\pi_{12} = \pi_{21} = \delta$. Thus, the slope of firm 1's best reply is $2\beta/\delta$, the slope of firm 2's best reply is $\delta/2\beta$, and the stability conditions is $\pi_{11}\pi_{22} - \pi_{12}\pi_{21} = 4\beta^2 - \delta^2 > 0$. Thus, the model is stable when $\beta > \delta/2$.

Figure 9.1 Best-Reply Functions and the Cournot Equilibrium

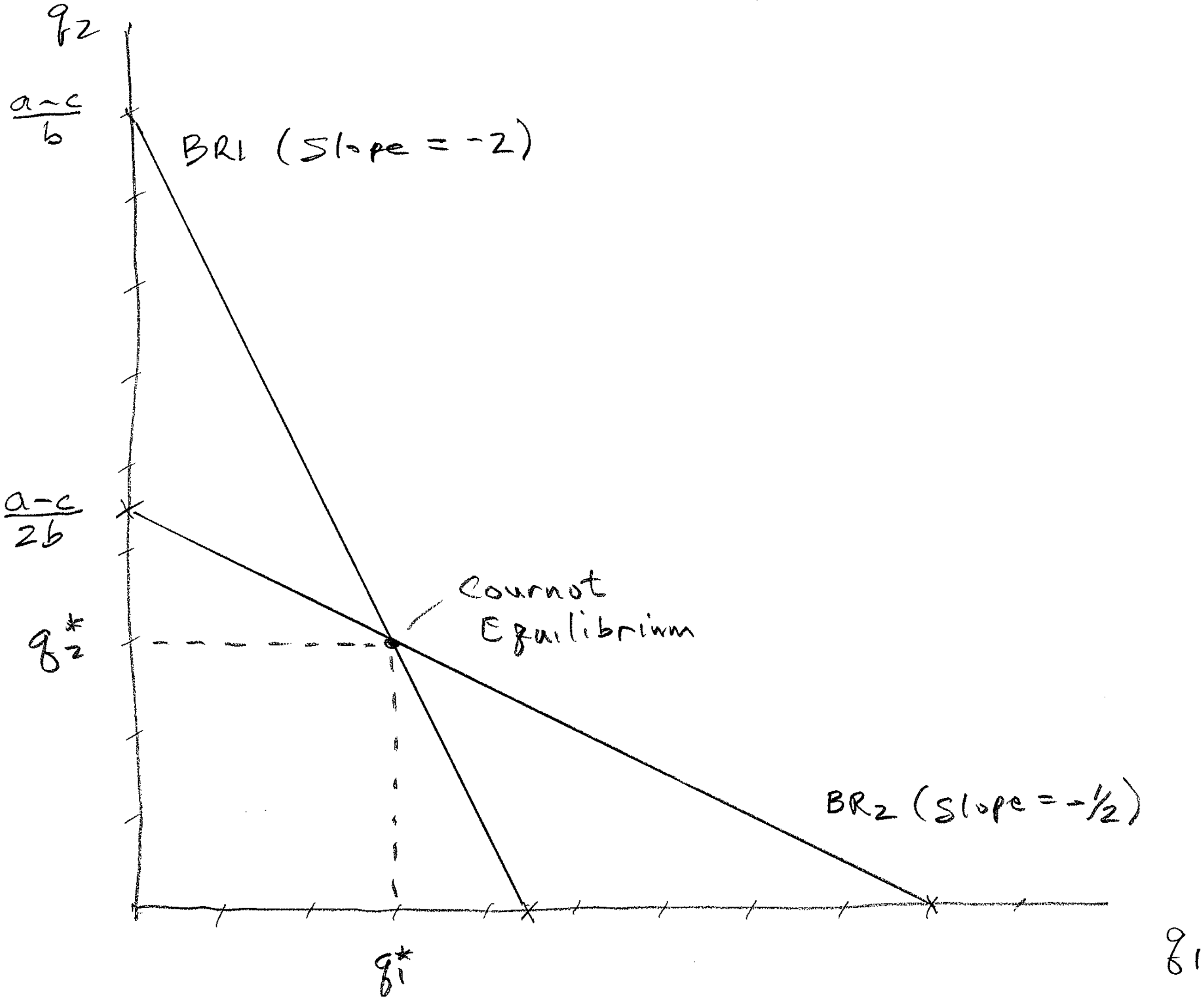


Figure 9.2 Firm 1's Isoprofits

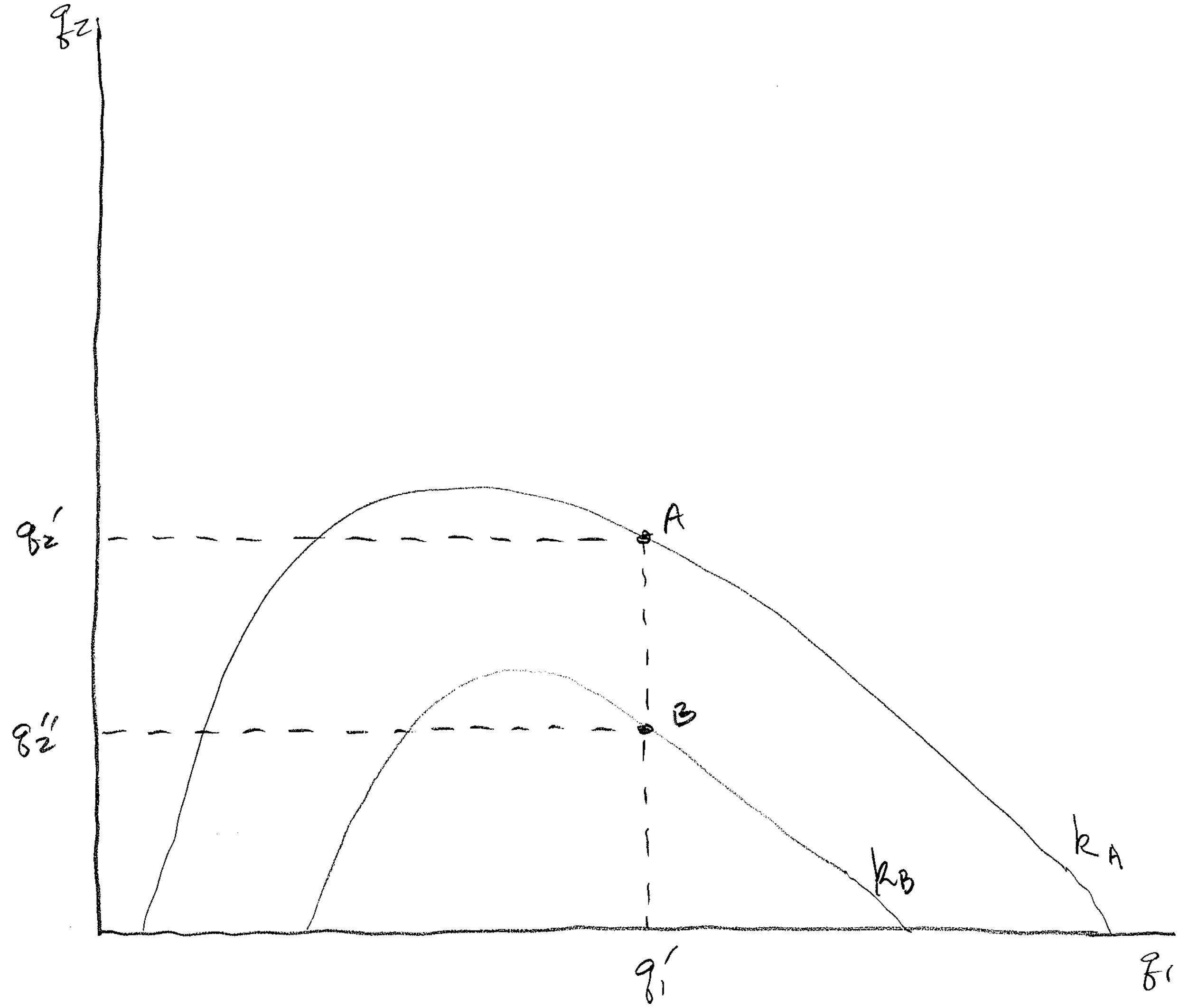


Figure 9.3 Derivation of Firm 1's Best-Reply Function

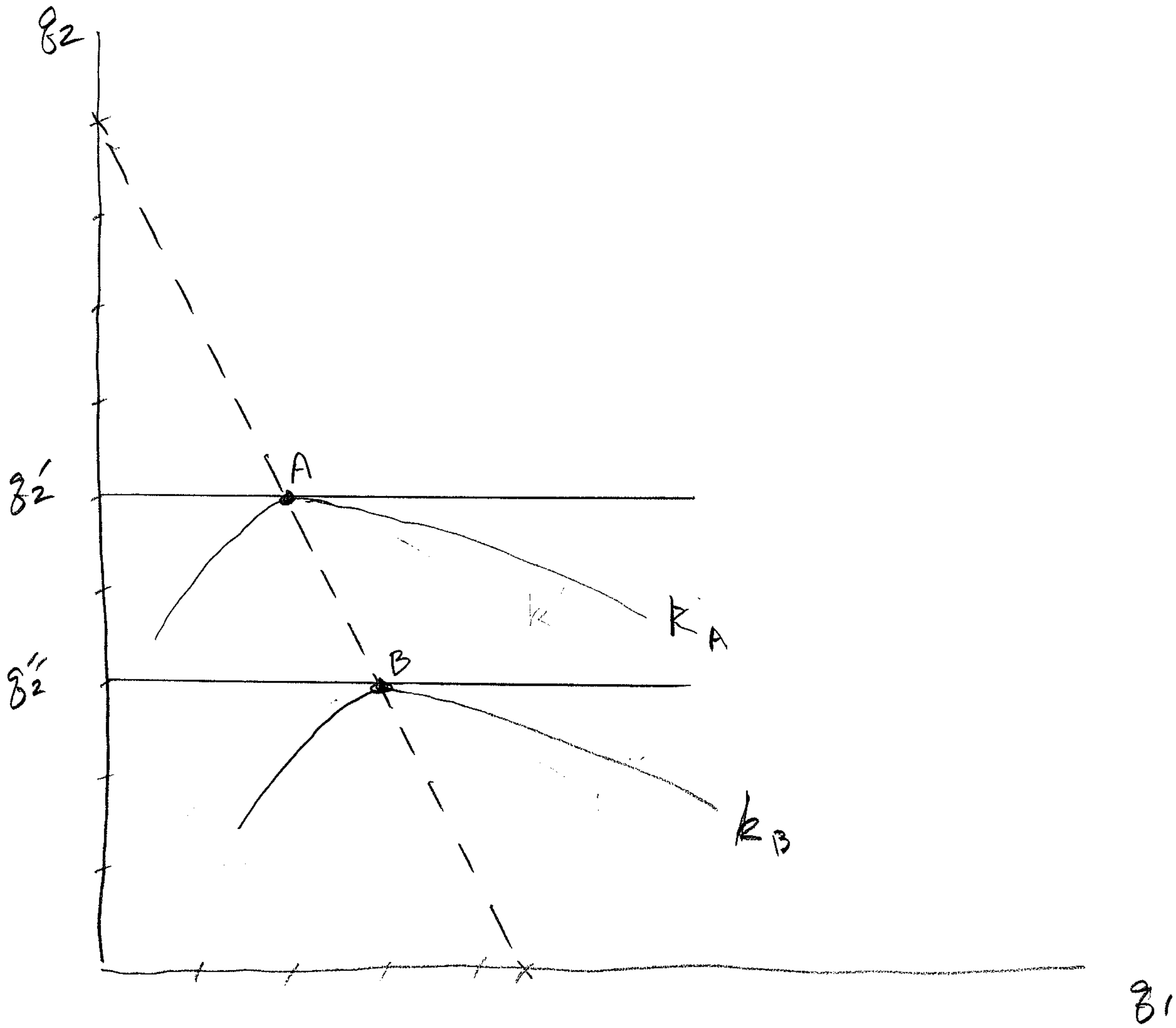


Figure 9.4 The Cournot Equilibrium with Best-Reply Functions and Isoprofits

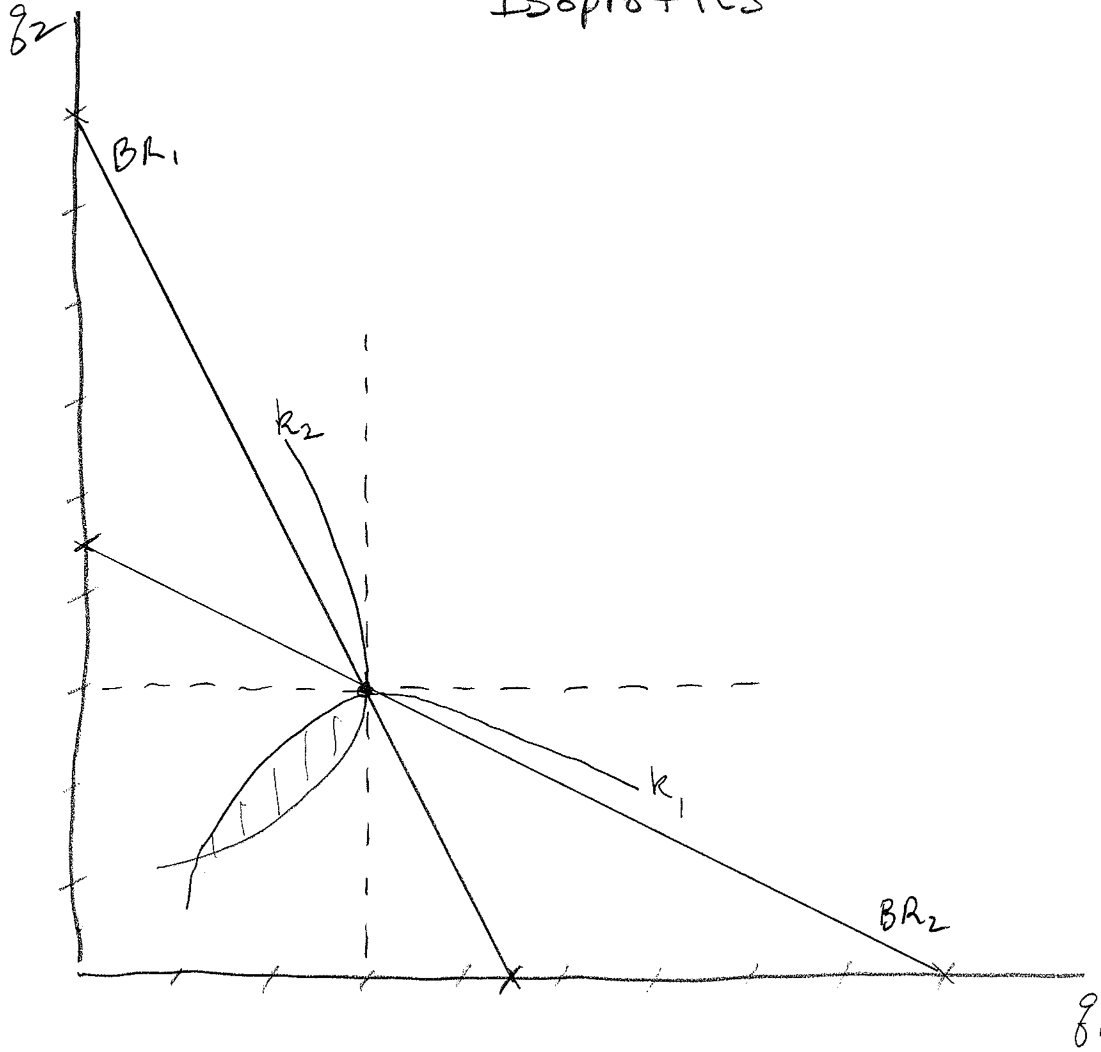


Figure 9.5 The Cournot Equilibrium when Firm 1 has Lower Costs than Firm 2

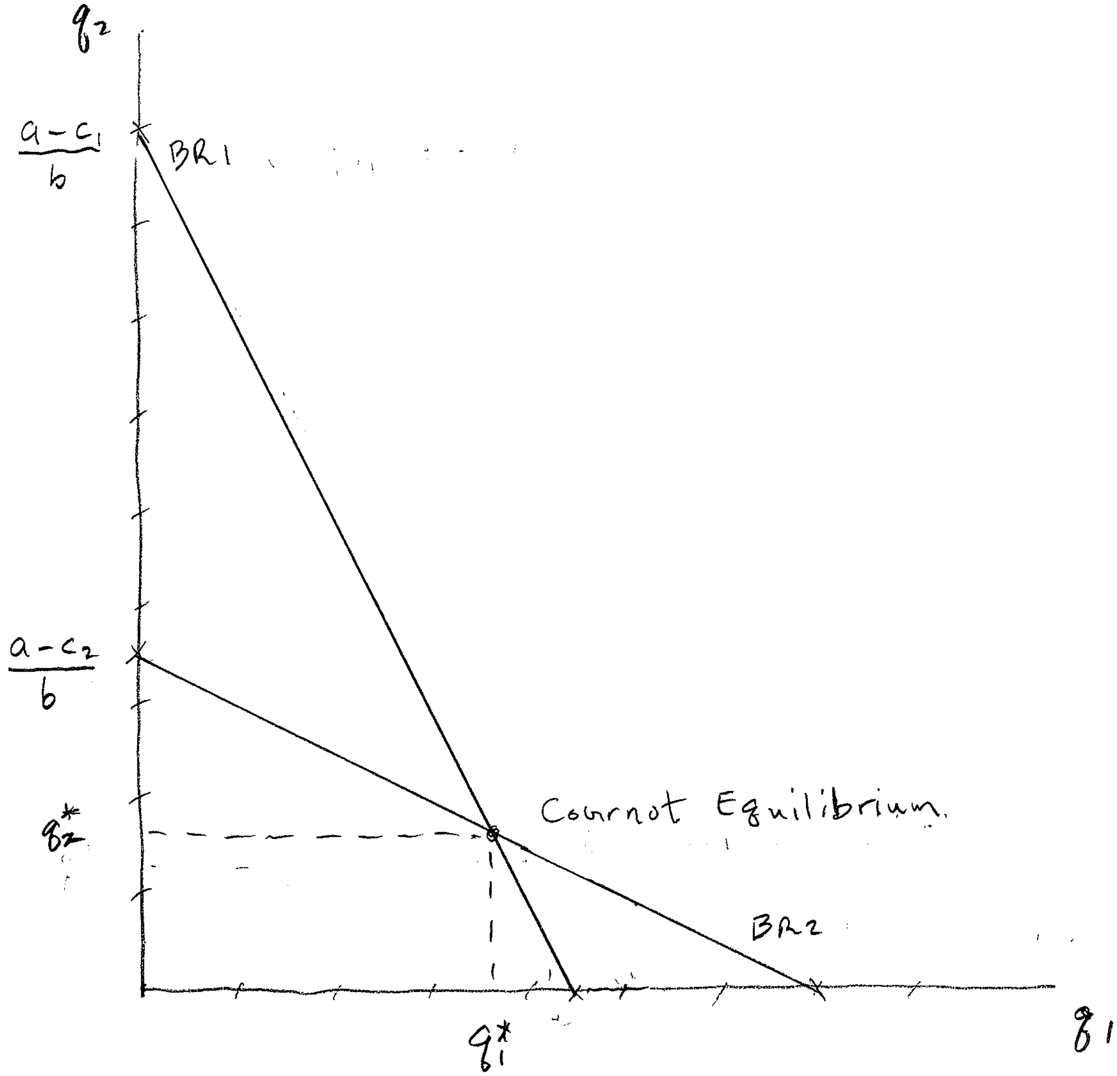


Figure 9.6 The Cournot Equilibrium When Firm 2 Shuts Down, Leaving Firm 1 in a Monopoly Position

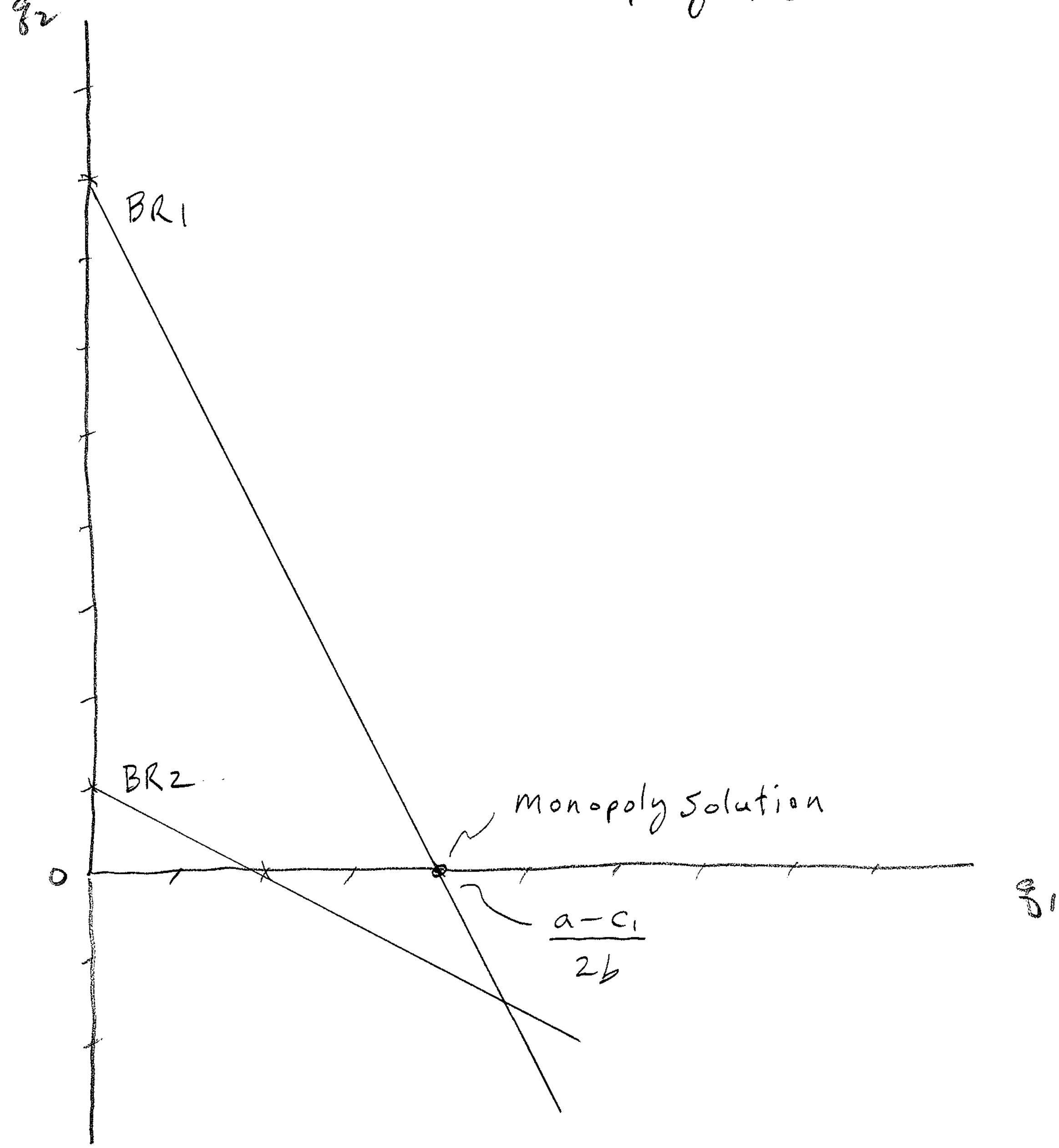


Figure 9.7 The Cournot Equilibrium and the Number of Competitors

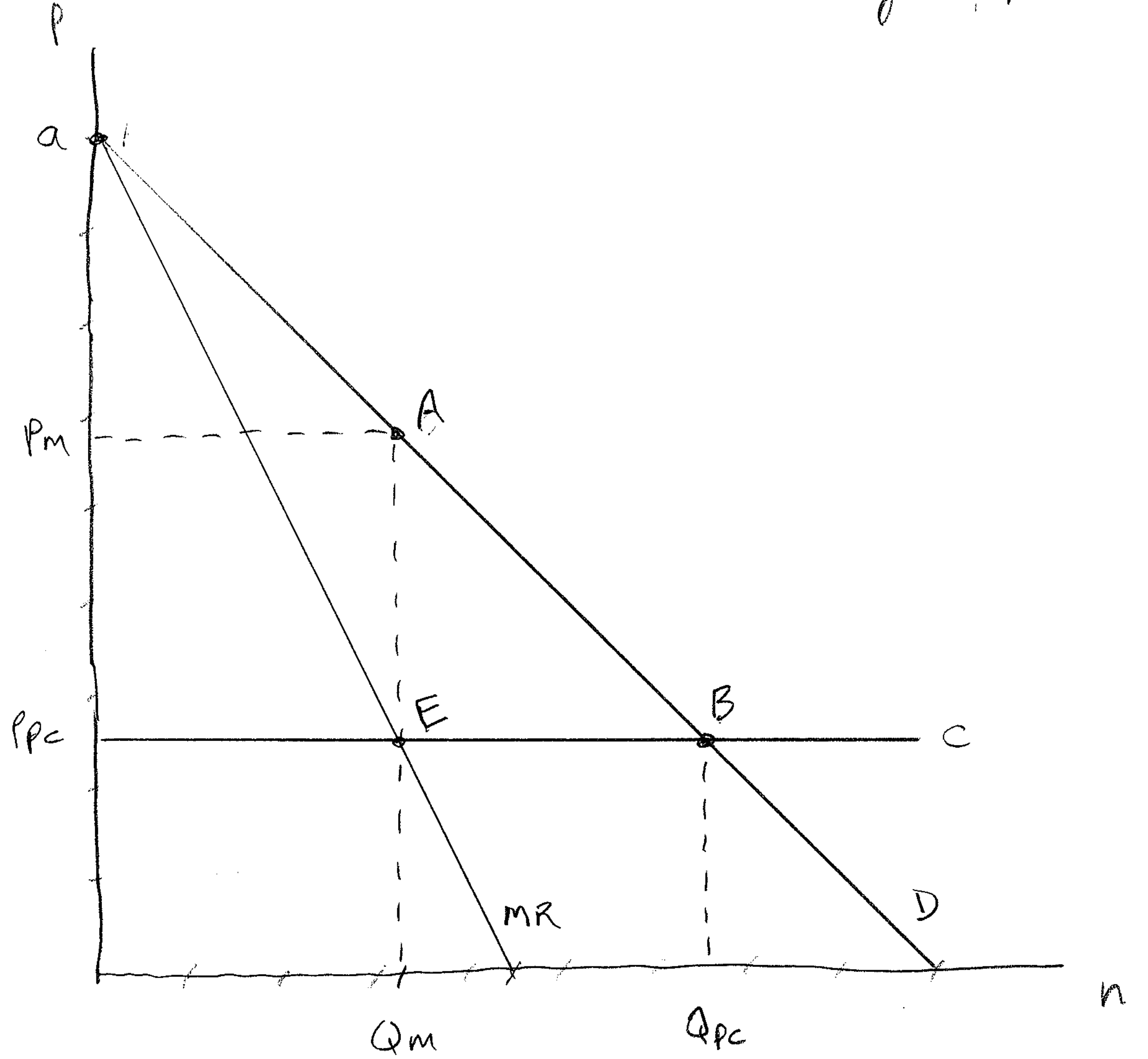


Figure 9.8   Firm i's Demand Function (d_i) in a Bertrand Game.

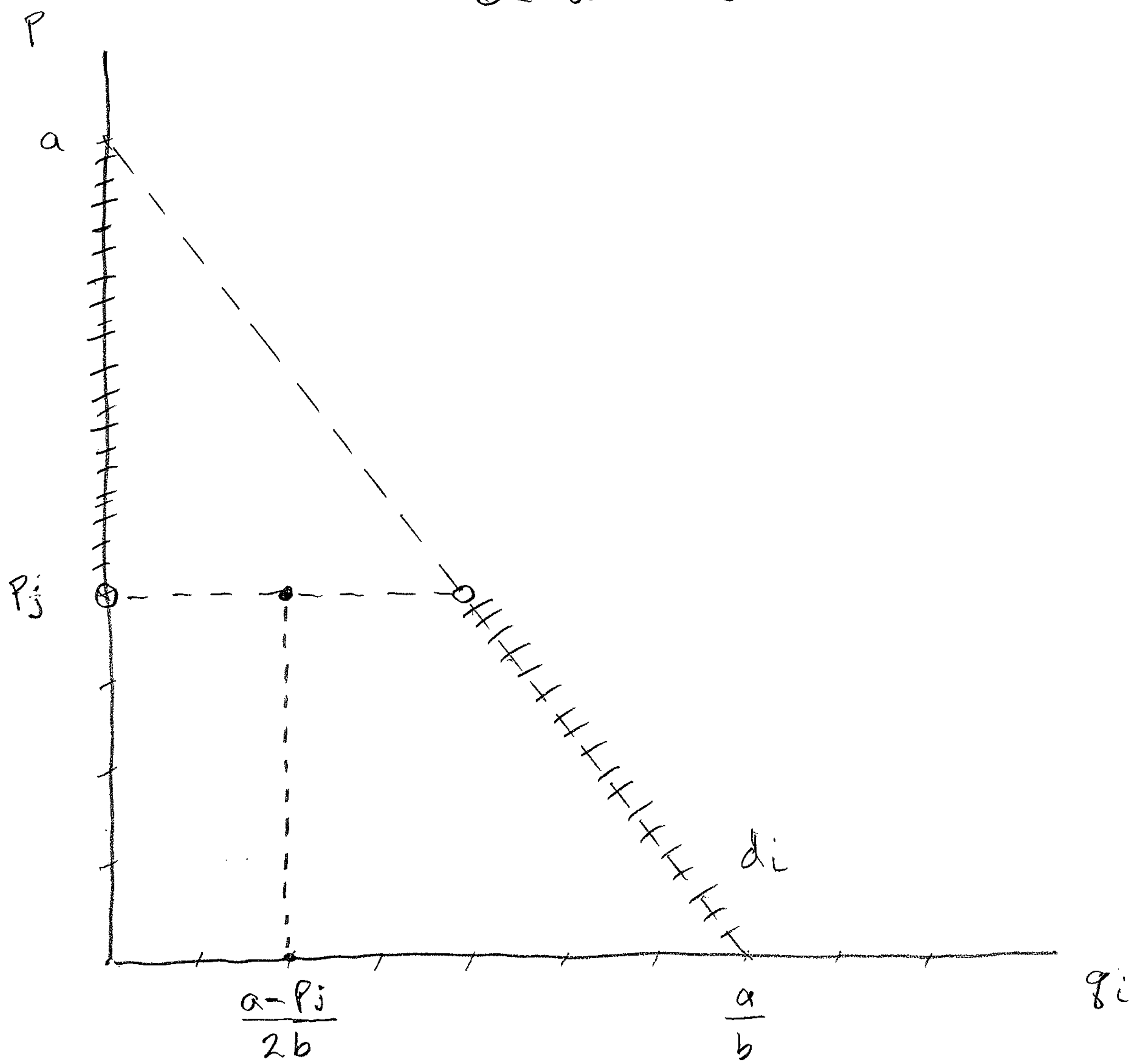


Figure 9.9 Firm i's Profits in a Bertrand Game

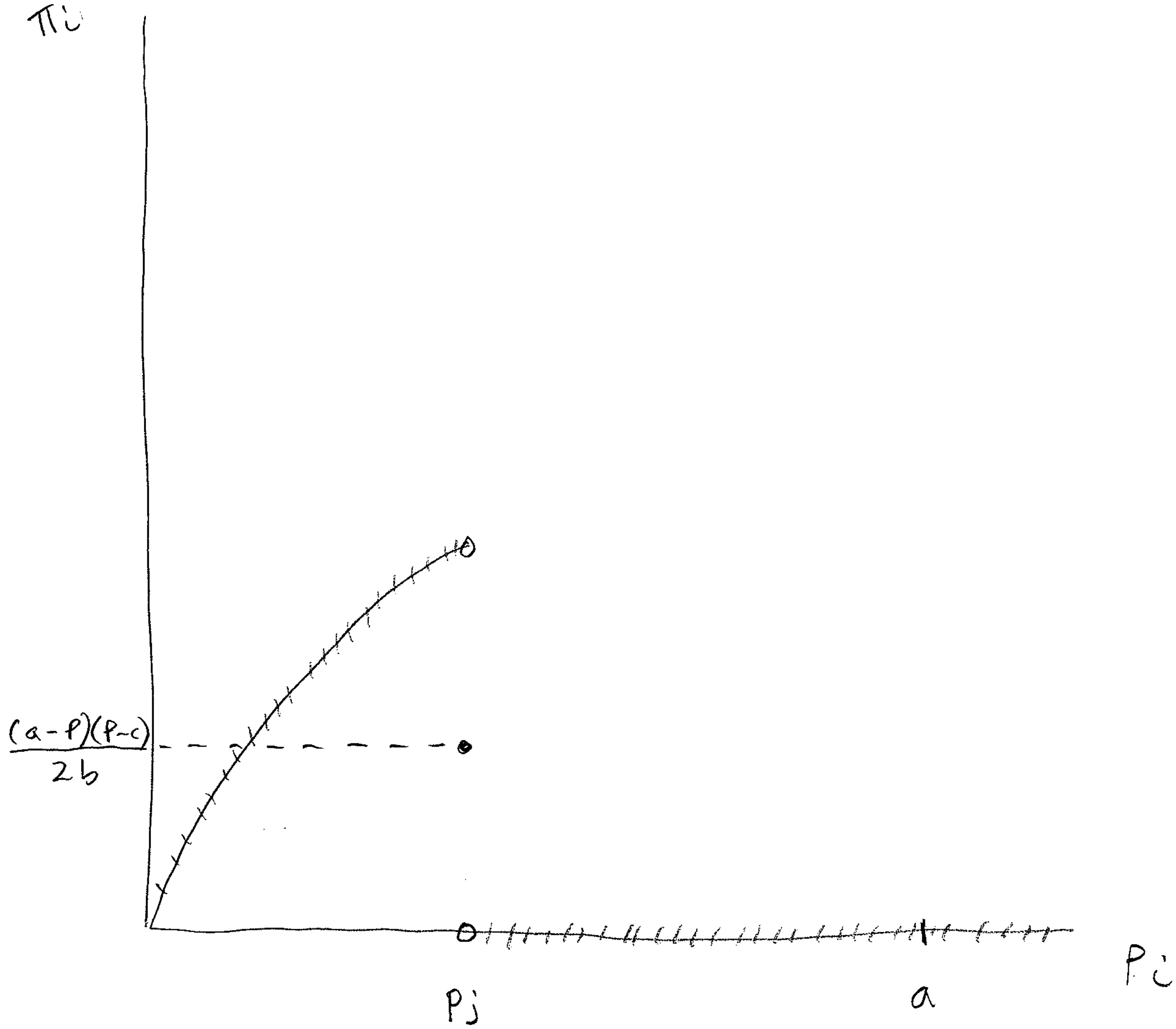


Figure 9.10 The Cournot Equilibrium with Product Differentiation

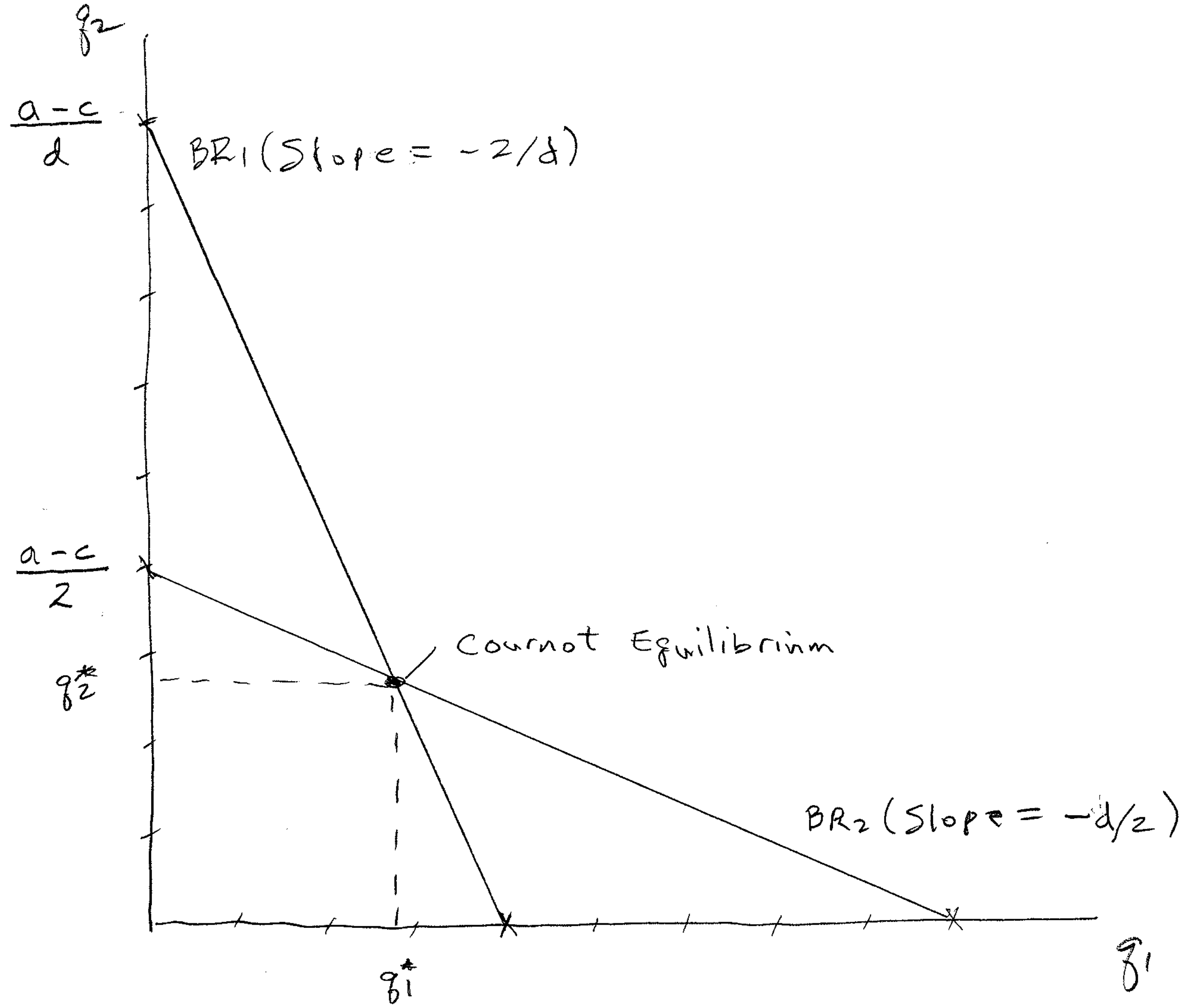
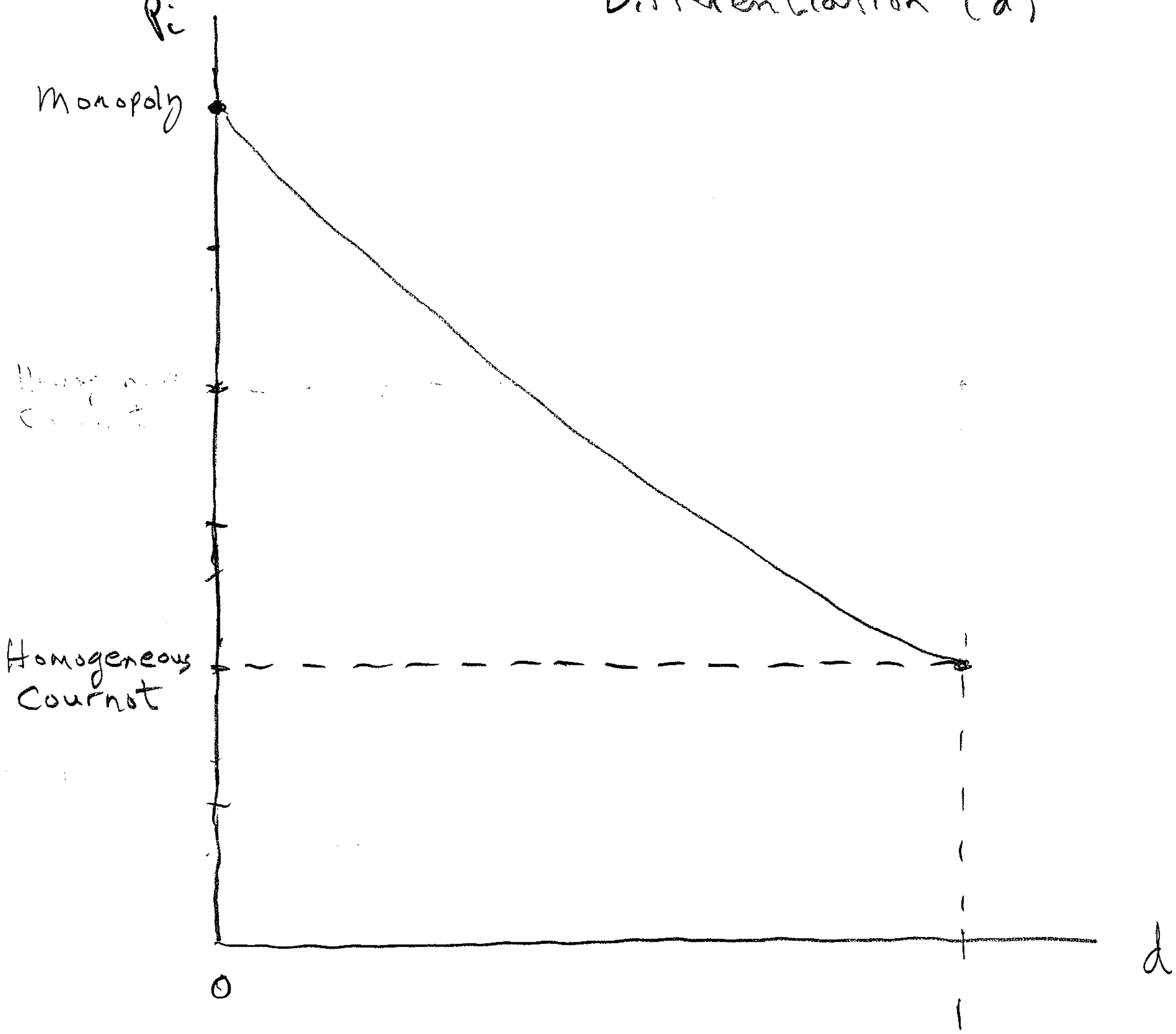


Figure 9.11 The Cournot Equilibrium Price for Different Levels of Product Differentiation (d)



Note:
a=b
c=0
b=1

Figure 9.12 The Bertrand Equilibrium with Product Differentiation

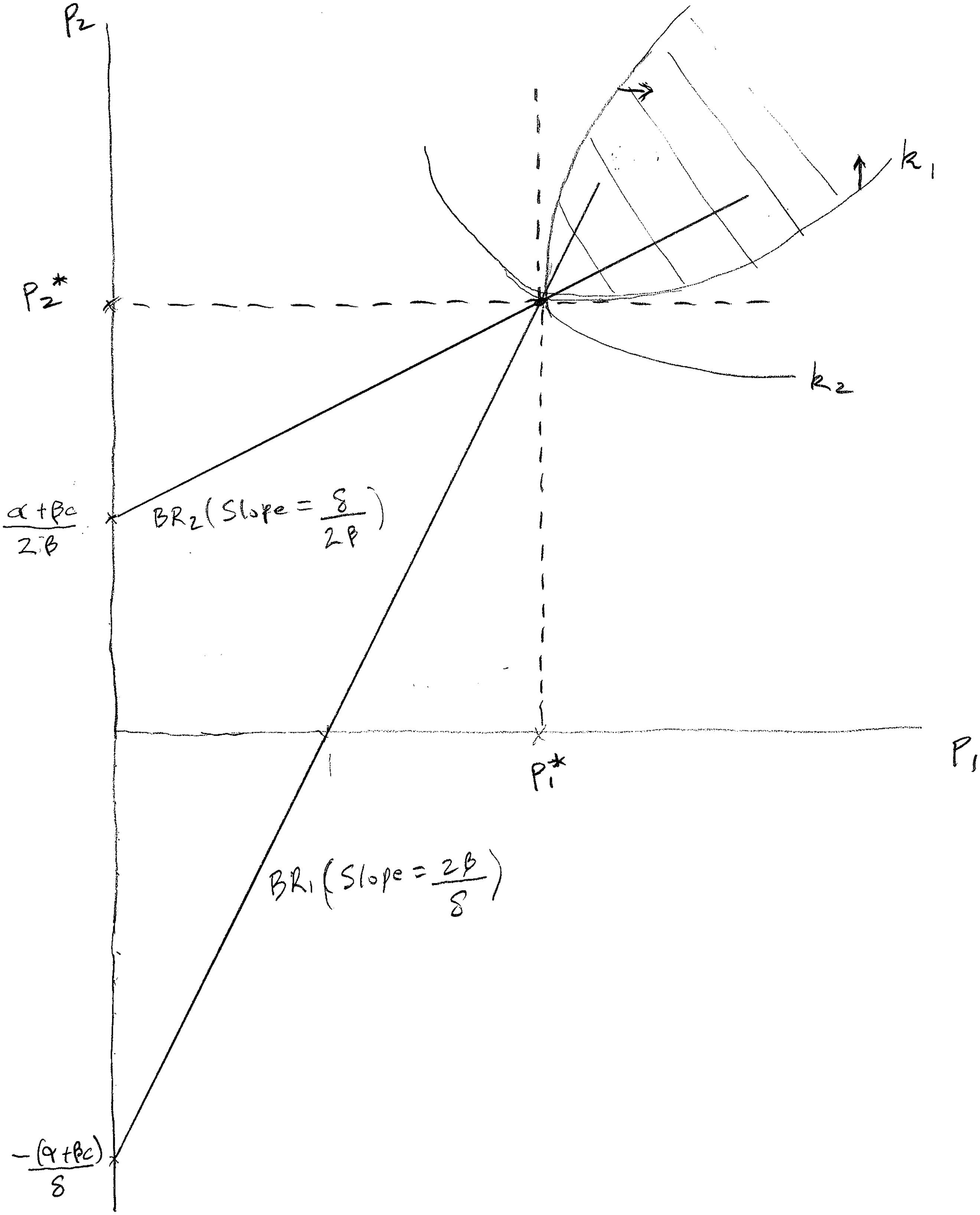


Figure 9.13 The Bertrand Equilibrium with Vertical Differentiation

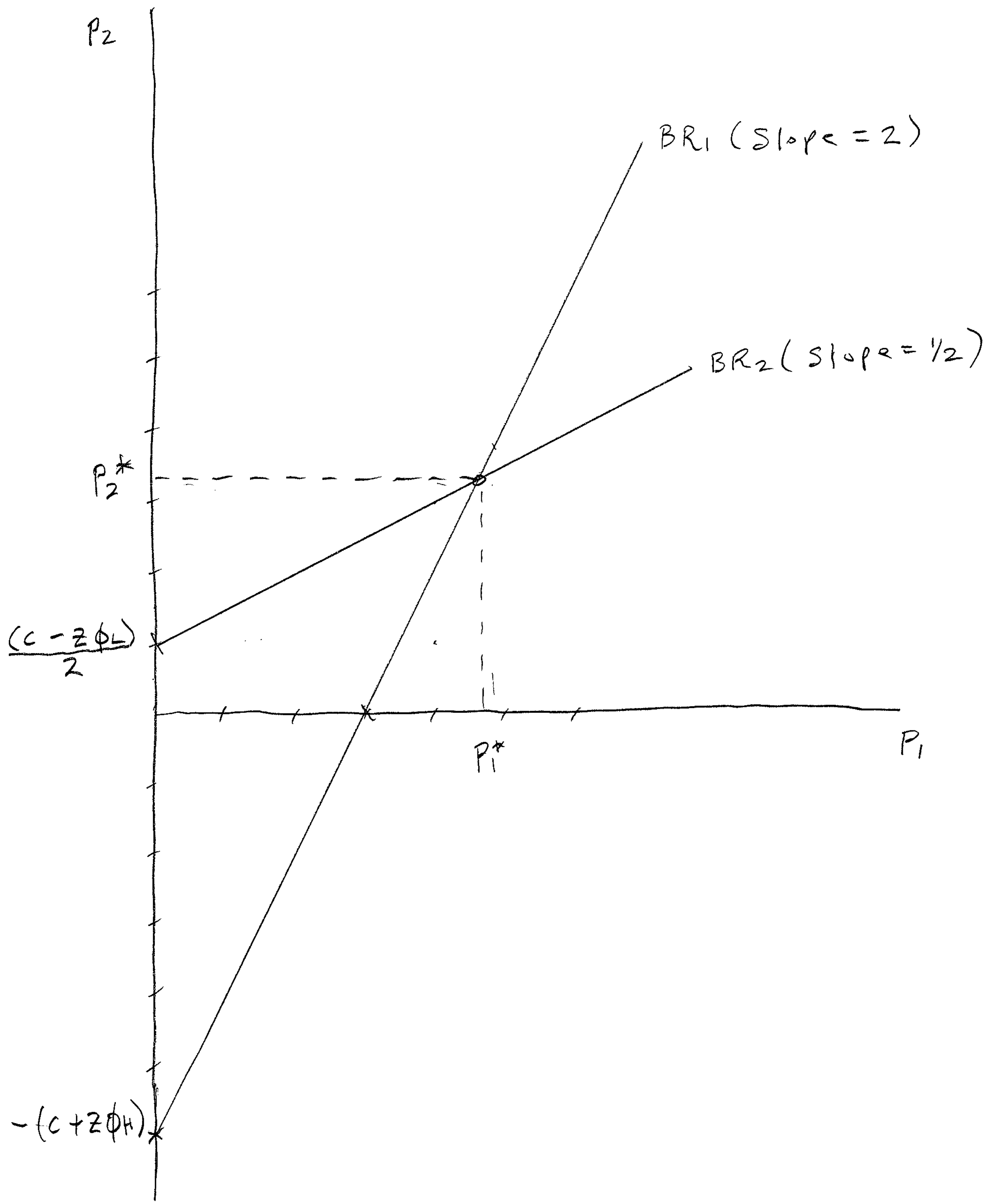


Figure 9.14 The Cournot-Bertrand Equilibrium

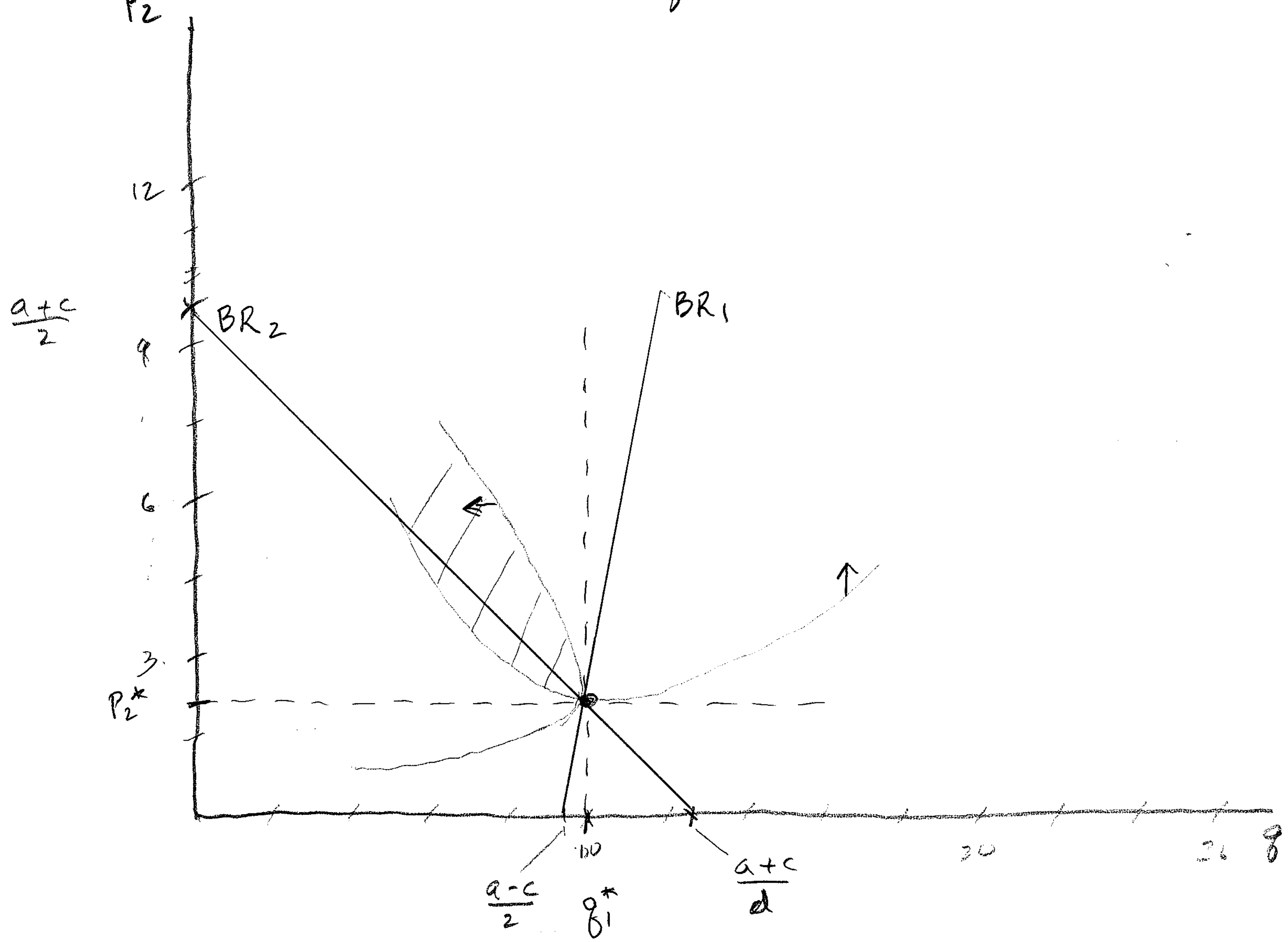


Figure 9.15 A Stable Cournot Model

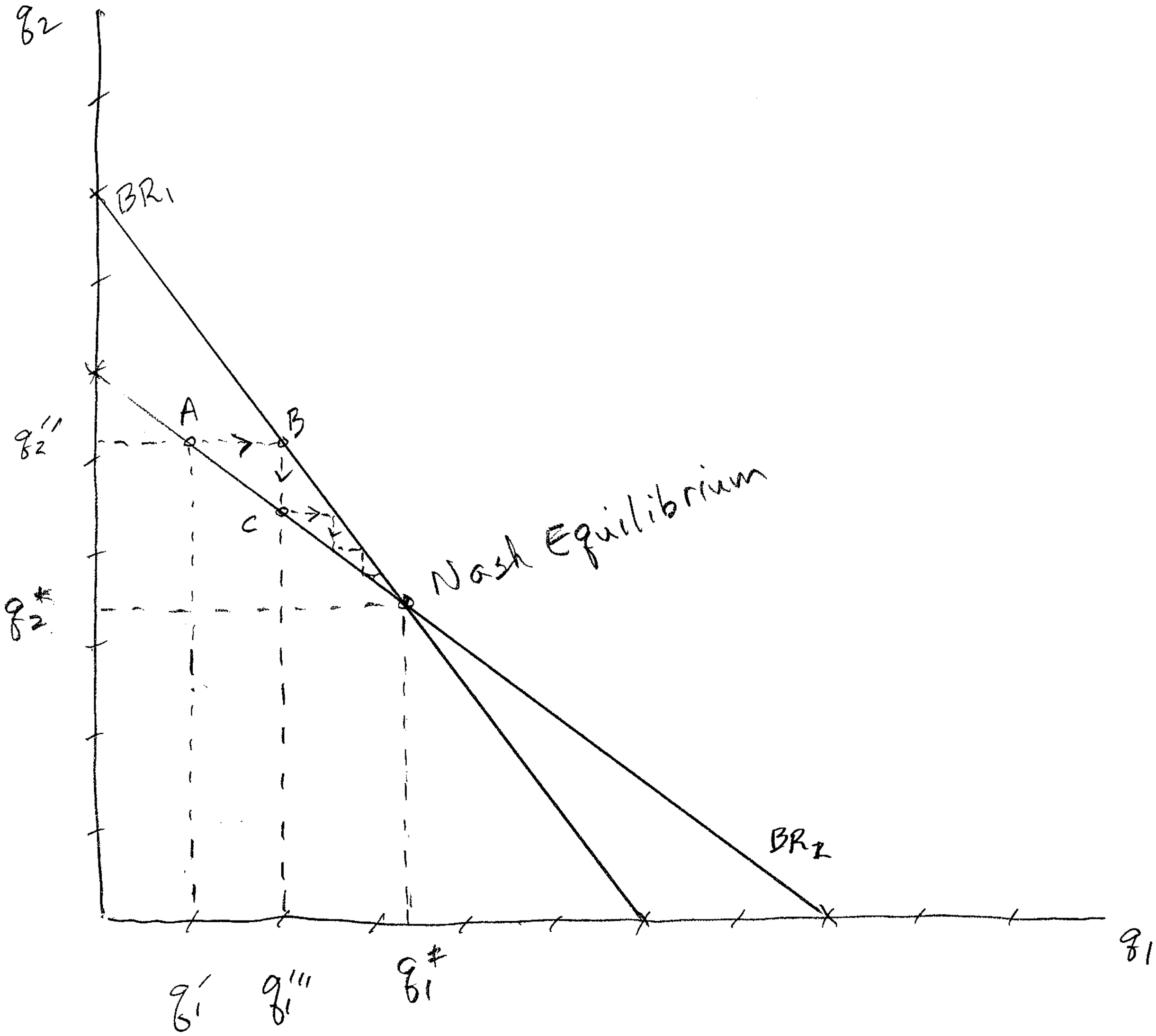


Figure 9.16 An Unstable Cournot Model

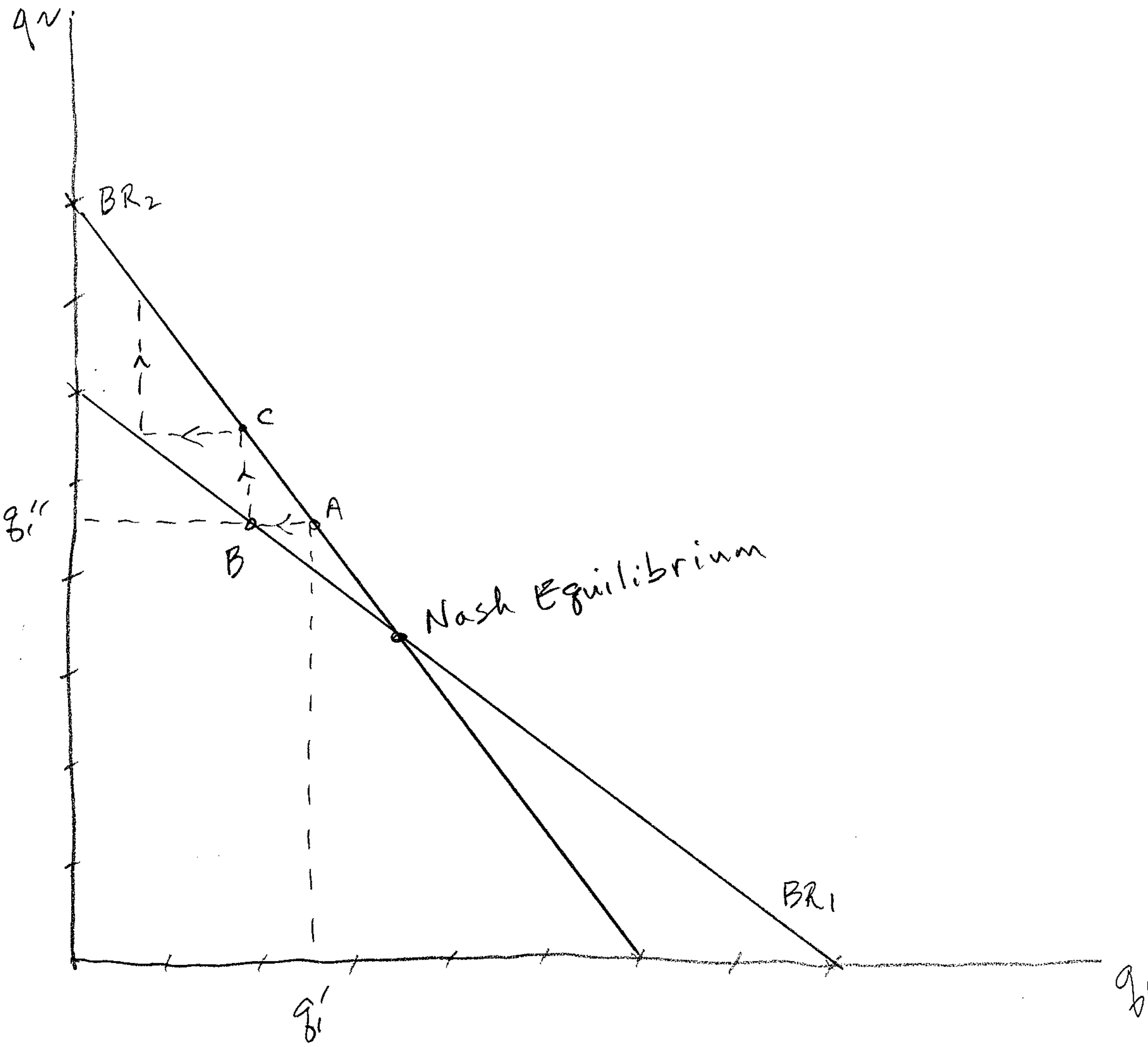


Figure 9.17 A Stable Bertrand Model

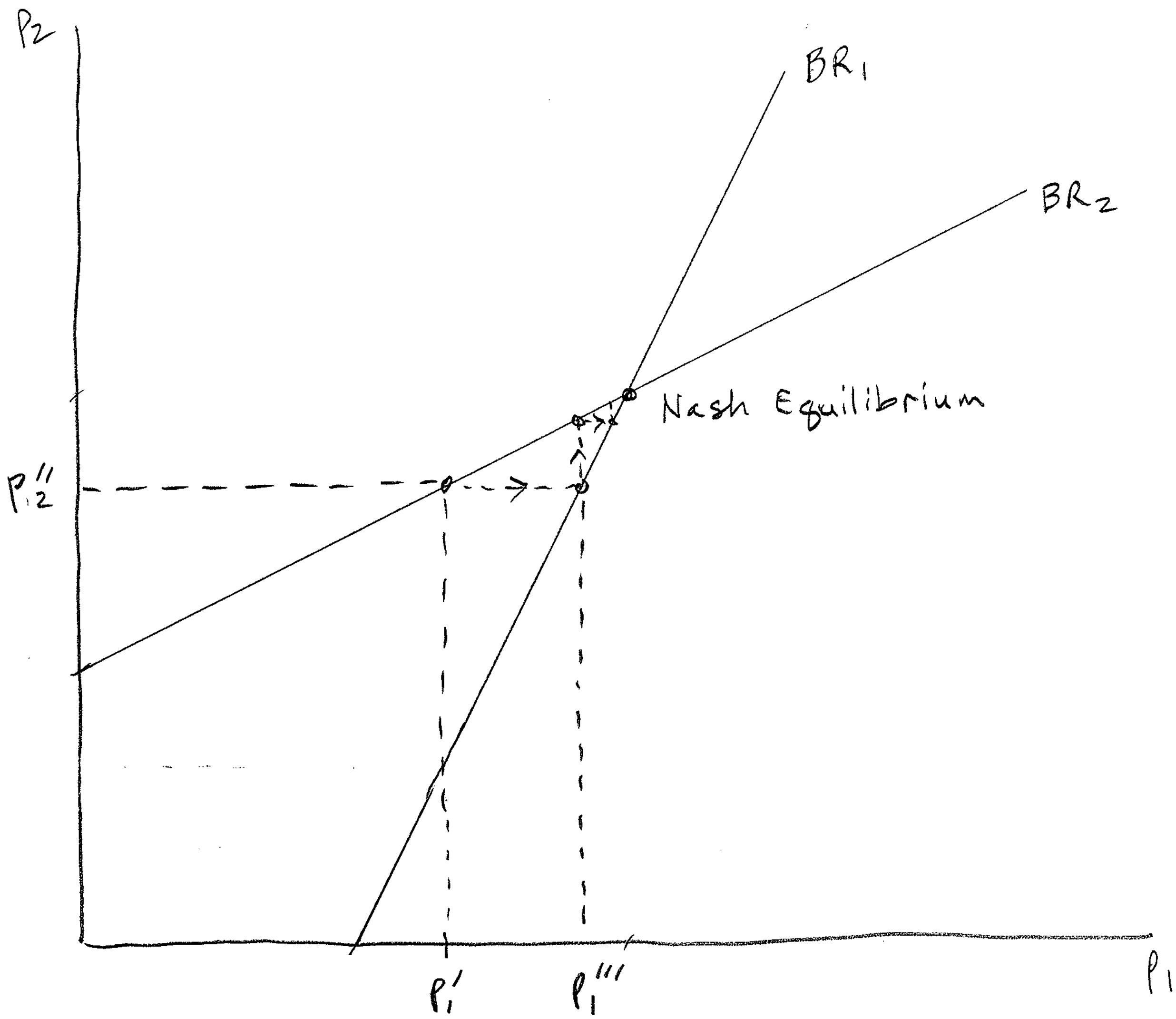


Table 9.1 Nash Equilibrium Profits for Cournot (C), Bertrand (B), Cournot-Bertrand (CB), and Bertrand-Cournot (BC) Outcomes

Firm 1's Strategic Variable	Firm 2's Strategic Variable: q_2	Firm 2's Strategic Variable: p_2
q_1	$\pi_1^C = 101.7 - F_1^C$, $\pi_2^C = 14.4 - F_2^C$	$\pi_1^{CB} = 100 - F_1^C$, $\pi_2^{CB} = 4 - F_2^B$
p_1	$\pi_1^{BC} = 88 - F_1^B$, $\pi_2^{BC} = 11.2 - F_2^C$	$\pi_1^B = 97.6 - F_1^B$, $\pi_2^B = 4.04 - F_2^B$