This activity is worth 20 points of course credit. See tentative calendar for due dates. Late activities are accepted at the discretion of your recitation instructor and a penalty may be imposed.

(1) Each of the following is an invalid proof of the proposition below. In each instance, clearly explain what the logical error is (with complete and grammatically correct sentences).

∀n ∈ ℤ, if n^2 is even then n is even.

(a) (2 pts) Proof: Let n ∈ ℤ. Assume n^2 is even. Then ∃k ∈ ℤ, n^2 = 2k. Then 
   \[ n = \sqrt{2k} = 2\ell \text{ for some } \ell \in ℤ. \] Hence n is even □

(b) (2 pts) Proof: Let n ∈ ℤ. Assume n^2 is even. To yield a contradiction, assume n 
   is not even. Then n is odd and so ∃k ∈ ℤ, n = 2k + 1. But then 
   \[ n^2 = (2k+1)^2 = 4k^2 + 1 = 2(2k^2 + 1). \] Since 2k^2 ∈ ℤ it would follow that n^2 is odd, which give 
   us a contradiction □

(c) (2 pts) Proof: Let n ∈ ℤ. Assume n is even. Then ∃k ∈ ℤ, n = 2k. Then 
   \[ n^2 = (2k)^2 = 4k^2 = 2(2k^2). \] Since 2k^2 ∈ ℤ it follows that n^2 is even □

(d) (2 pts) Proof: Let n ∈ ℤ. Assume n^2 is even. To yield a contradiction, assume n 
   is not even. Then n is odd and so ∃k such that n = 2k + 1. But then 
   \[ n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1. \] Hence n^2 is odd, which give 
   us a contradiction □
(2) (2 pts) *Begging the Question*: A fallacy in which the premise of an argument presupposes the truth of its conclusion; in other words, the argument takes for granted (as true) what it is supposed to prove.

Recall: \([x] \) is the “floor function” (greatest integer less than or equal to \(x\)).

Consider the following invalid proof that begs the question:

\[
\forall x \in \mathbb{R}, \ [2x - 1] = [2x] - 1.
\]

*Proof*: Let \( x \in \mathbb{R} \). Set \([2x] = k\). Then by subtracting one from both sides,

\[
[2x - 1] = k - 1.
\]

Then we substitute \( k = [2x] \) to get

\[
[2x - 1] = [2x] - 1\]

Explain exactly how this proof begs the question? Please use complete and grammatically correct sentences!
(3) Both of the following are valid proofs for the proposition below. In each instance, identify the proof method used.

∀n ∈ ℝ, ∀m ∈ ℕ, if n^m is irrational then n is irrational.

(a) (1 pts) Proof: Let n ∈ ℝ and m ∈ ℕ. Assume n ∈ ℚ. Then ∃p, q ∈ ℤ, q ≠ 0 and n = p/q. Then n^m = p^m/q^m. Since p^m, q^m ∈ ℤ and q^m ≠ 0 it follows that n^m ∈ ℚ which completes the proof □

(b) (1 pts) Proof: Let n ∈ ℝ and m ∈ ℕ. Assume n^m is irrational. Suppose n ∈ ℚ. Then ∃p, q ∈ ℤ, q ≠ 0 and n = p/q. Then n^m = p^m/q^m. Since p^m, q^m ∈ ℤ and q^m ≠ 0 it would follow that n^m ∈ ℚ □

(4) (2 pts) Can any proof by contrapositive be swapped out for a proof by contradiction? Can any proof by contradiction be swapped out for a proof by contrapositive? Explain (with complete and grammatically correct sentences).
(5) (2 pts) Write a careful proof of the following true proposition:

\[
\forall x, y \in \mathbb{R}, \text{ if } xy \text{ is irrational then either } x \text{ is irrational or } y \text{ is irrational.}
\]

\textit{Proof:}
(6) Prove or disprove:

(a) (2 pts) For any sets $X, Y$, $\mathcal{P}(X \cup Y) \subseteq \mathcal{P}(X) \cup \mathcal{P}(Y)$.

(b) (2 pts) For any sets $X, Y$, $\mathcal{P}(X) \cup \mathcal{P}(Y) \subseteq \mathcal{P}(X \cup Y)$. 