MTH 306 - Lecture 6

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Topics:

- Lines and planes.
- Systems of linear equations.
- Systematic elimination of unknowns.
- Coefficient matrix. Augmented matrix.
- Determinants.
• **Lines.**

A **line** $L$ is determined by two points, $P_0$ and $P$. Alternatively line $L$ can be described by point $P_0$ and direction vector $\vec{v}$. 
• Parametric equation.

Given a line going through point $P_0$ in the direction $\vec{v}$ it can be described as the following parametric equation:

$$\vec{r} = \vec{r}_0 + t\vec{v}, \quad \text{for } -\infty < t < \infty,$$

where $\vec{r}_0$ is the vector from the origin to $P_0$. 
• Parametric equation.

**Parametric equation:** \( \vec{r} = \vec{r}_0 + t\vec{v} \)

• **Example:** 2D. Given \( \vec{r}_0 = (x_0, y_0) \) and \( \vec{v} = (a, b) \), then

\[
(x, y) = (x_0, y_0) + t(a, b) = (x_0 + at, y_0 + bt)
\]
• **Example: 2D.** Given $\vec{r}_0 = (x_0, y_0)$ and $\vec{v} = (a, b)$, then

$$(x, y) = (x_0, y_0) + t(a, b) = (x_0 + at, y_0 + bt)$$

gives

$$\begin{align*}
  x &= x_0 + at \\
  y &= y_0 + bt
\end{align*}$$

for $-\infty < t < \infty$.

These equations are called **(scalar) parametric equations** for the line $L$.

The parametric equations for lines can be obtained similarly in 3-D, and higher dimensions. See the book.
• **Example:** Find parametric equations for the line $L$ determined by the two points, $(2, 5)$ and $(1, 7)$.

**Solution:** Take $\vec{r}_0 = (2, 5)$ (as $P_0 = (2, 5)$ is a point on $L$), and

$$
\vec{v} = (1 - 2, 7 - 5) = (-1, 2)
$$

The parametric equation $\vec{r} = \vec{r}_0 + t\vec{v}$ reads

$$
\begin{cases}
  x = 2 - t \\
  y = 5 + 2t
\end{cases}
$$
• **Planes in 3-D.**

A plane $\Pi$ in 3-D space $\mathbb{R}^3$ is determined by a point $P_0 = (x_0, y_0, z_0)$ on it and a vector $\vec{n}$ perpendicular (normal) to $\Pi$. 
• Planes in 3-D.

Every point \((x, y, z)\) on the plane \(\Pi\) has to satisfy

\[
(x - x_0, y - y_0, z - z_0) \perp \vec{n}
\]

In other words, \((x - x_0, y - y_0, z - z_0) \cdot \vec{n} = 0\),
which we write as

\[
a(x - x_0) + b(y - y_0) + c(z - z_0) = 0
\]
Equation \( a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \)

- **Example:** Find an equation for the plane in 3-D that contains point \((1, 0, 2)\) and is perpendicular to vector \(\vec{n} = (4, 2, -1)\).

**Answer:** We use \((x - x_0, y - y_0, z - z_0) \cdot \vec{n} = 0\) with \((x_0, y_0, z_0) = (1, 0, 2)\), obtaining

\[
4(x - 1) + 2(y - 0) - (z - 2) = 0
\]

which we simplify to

\[
4x + 2y - z - 2 = 0
\]
• **Systems of linear equations.**

Consider a **system** of two **linear equations**:

\[
\begin{align*}
  x - 2y &= 0 \\
  2x + 3y &= 14
\end{align*}
\]

To solve it means to find all \((x, y)\) that satisfy both equations.

**Solution:** use the first equation to express \(x\) via \(y\), obtaining \(x = 2y\). Then plug \(x = 2y\) into the second equation: \(2(2y) + 3y = 14\). Thus \(7y = 14\). So, \(y = 2\) and \(x = 2y = 4\).

**Answer:** \(x = 4, y = 2\)
• **Systems of linear equations.**

\[
\begin{align*}
  x - 2y &= 0 \quad (L_1) \\
  2x + 3y &= 14 \quad (L_2)
\end{align*}
\]

To solve it means to find all \((x, y)\) that satisfy both equations.
• Systems of linear equations.

\[
\begin{align*}
ax + by &= e \quad (L_1) \\
(cx + dy &= f \quad (L_2)
\end{align*}
\]

**Case I.** $L_1$ and $L_2$ intersect in a **unique** point.
• Systems of linear equations.

\[
\begin{align*}
ax + by &= e \quad (L_1) \\
\quad cx + dy &= f \quad (L_2)
\end{align*}
\]

**Case II.** \(L_1\) and \(L_2\) are distinct parallel lines.
• Systems of linear equations.

\[
\begin{align*}
ax + by &= e \quad (L_1) \\
fx + dy &= f \quad (L_2)
\end{align*}
\]

Case III. $L_1$ and $L_2$ coincide.
• **Systematic elimination of unknowns.**

\[
\begin{align*}
\begin{cases}
  x - 2y &= 0 \quad Eq.1 \\
  2x + 3y &= 14 \quad Eq.2 \\
\end{cases}
\end{align*}
\]

\[
\downarrow
\]

\[
\begin{align*}
\begin{cases}
  x - 2y &= 0 \quad Eq.1 \\
  7y &= 14 \quad Eq.2 \rightarrow 2 \times Eq.1 \rightarrow Eq.2 \\
\end{cases}
\end{align*}
\]

\[
\downarrow
\]

\[
\begin{align*}
\begin{cases}
  x - 2y &= 0 \quad Eq.1 \\
  y &= 2 \quad \frac{1}{7}Eq.2 \rightarrow Eq.2 \\
\end{cases}
\end{align*}
\]

\[
\downarrow
\]

\[
\begin{align*}
\begin{cases}
  x &= 4 \quad Eq.1 \quad + \quad 2 \times Eq.2 \\
  y &= 2 \quad Eq.2 \\
\end{cases}
\end{align*}
\]
<table>
<thead>
<tr>
<th>System of Equations</th>
<th>Augmented Matrix</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\begin{cases} x - 2y = 0 \ 2x + 3y = 14 \end{cases}$</td>
<td>$\begin{bmatrix} 1 &amp; -2 &amp; 0 \ 2 &amp; 3 &amp; 14 \end{bmatrix}$ $R1$ $R2$</td>
</tr>
<tr>
<td>$\rightarrow$</td>
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</tr>
</tbody>
</table>
• **Coefficient matrix. Augmented matrix.** Consider a system of two linear equations with two unknowns:

\[
\begin{align*}
ax + by &= e \\
 cx + dy &= f
\end{align*}
\]

Group the coefficients of the unknowns into a rectangular array called the **coefficient matrix**

\[
A = \begin{bmatrix}
a & b \\
  c & d
\end{bmatrix}
\]

The system is fully described by its **augmented matrix**

\[
\begin{bmatrix}
 a & b & e \\
  c & d & f
\end{bmatrix}
\]
- **Theorem.** An $n \times n$ linear system of algebraic equations either has a unique solution, no solution, or infinitely many solutions.

\[
\begin{align*}
& a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n = b_1 \\
& a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n = b_2 \\
& \quad \vdots \\
& a_{n,1}x_1 + a_{n,2}x_2 + \cdots + a_{n,n}x_n = b_n
\end{align*}
\]
**Example.**

\[
\begin{align*}
\begin{cases}
  x + 2y &= 1 \\
  2x - y &= 7
\end{cases} &\quad \begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 7 \end{bmatrix} \quad R1 \\
\downarrow &\quad \downarrow \\
\begin{cases}
  x + 2y &= 1 \\
  -5y &= 5
\end{cases} &\quad \begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & 5 \end{bmatrix} \quad R2 - 2R1 \\
\downarrow &\quad \downarrow \\
\begin{cases}
  x + 2y &= 1 \\
  y &= -1
\end{cases} &\quad \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \quad \frac{1}{5}R2 \\
\downarrow &\quad \downarrow \\
\begin{cases}
  x &= 3 \\
  y &= -1
\end{cases} &\quad \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \end{bmatrix} \quad R1 - 2R2
\end{align*}
\]
\[
\begin{align*}
\begin{cases}
  x + 2y = 1 \\
  2x - y = 7 
\end{cases}
\end{align*}
\Rightarrow
\begin{align*}
\begin{cases}
  x = 3 \\
  y = -1 
\end{cases}
\end{align*}
\]

Here \( x + 2y = 1 \) and \( 2x - y = 7 \) intersect in a unique point \((3, -1)\).
• Example.
\[
\begin{align*}
\begin{cases}
x + 2y &= 1 \\
3x + 6y &= 3
\end{cases}
\end{align*}
\]
\[
\begin{align*}
\downarrow & \quad \downarrow \\
\begin{cases}
x + 2y &= 1 \\
0 &= 0
\end{cases}
\end{align*}
\]

Here \( x + 2y = 1 \) and \( 3x + 6y = 3 \) coincide.

Thus there are **ininitely many solutions**. The solutions line \( x + 2y = 1 \) can be expresses as a *parametric equation*:

\[
(x, y) = (1 - 2t, t) = (1, 0) + (-2, 1)t, \quad \text{where } y = t
\]
\[
\begin{align*}
\begin{cases}
x + 2y &= 1 \\
3x + 6y &= 3
\end{cases} 
\Rightarrow x + 2y = 1
\end{align*}
\]

Here \( x + 2y = 1 \) and \( 3x + 6y = 3 \) coincide.

\[
(x, y) = (1 - 2t, t) = (1, 0) + (-2, 1)t, \quad \text{where } y = t
\]
• Example.

\[
\begin{align*}
\begin{aligned}
x + 2y &= 1 \\
x + 2y &= 4
\end{aligned}
\end{align*}
\]

Here \( x + 2y = 1 \) and \( x + 2y = 4 \) are distinct parallel lines. The system has no solutions. As otherwise 0 would be equal to 3.
\[
\begin{align*}
\begin{cases}
x + 2y &= 1 \\
x + 2y &= 4
\end{cases}
\quad \Rightarrow 
\begin{cases}
x + 2y &= 1 \\
0 &= 3
\end{cases}
\end{align*}
\]

Here \(x + 2y = 1\) and \(x + 2y = 4\) are distinct parallel lines. The system has no solutions.
Determinant of a $2 \times 2$ coefficient matrix.

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a coefficient matrix of a linear system, then its **determinant** is defined as

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Other notations: $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $|A|$, or $\det(A)$
Determinant of a $2 \times 2$ coefficient matrix.

Consider a system of two linear equations with two unknowns:
\[
\begin{aligned}
ax + by &= e \\
 cx + dy &= f
\end{aligned}
\]

\[
\begin{vmatrix}
a & b \\
 c & d
\end{vmatrix} \neq 0 \iff \text{there is a unique solution.}
\]

• Example. \[
\begin{aligned}
x - 2y &= 0 \\
2x + 3y &= 14
\end{aligned}
\]
was shown to have a unique solution $(x = 4, y = 2)$, and

\[
\begin{vmatrix}
1 & -2 \\
2 & 3
\end{vmatrix} = 3 + 4 = 7 \neq 0
\]
\[ \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0 \iff \text{there is a unique solution.} \]

- **Example.** \[ \begin{cases} x + 2y = 1 \\ 2x - y = 7 \end{cases} \] was shown to have a unique solution \((x = 3, y = -1)\), and

\[
\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} = -1 - 4 = -5 \neq 0
\]

- **Example.** \[ \begin{cases} x + 2y = 1 \\ 3x + 6y = 3 \end{cases} \] was shown to have infinitely many solutions, and

\[
\begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix} = 6 - 6 = 0
\]
\[
\begin{vmatrix}
  a & b \\
  c & d
\end{vmatrix} \neq 0 \iff \text{there is a unique solution.}
\]

- Example. \(\begin{cases} x + 2y = 1 \\ x + 2y = 4 \end{cases}\) was shown to have \textbf{no solutions}, and

\[
\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = 2 - 2 = 0
\]